Spillovers and growth in a local interaction model*

A. Cassar†- R. Nicolini‡

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Abstract

In this paper we aim at studying to what extent spillovers between firms may foster economic growth. The attention is addressed to the spillovers connected with the R&D activity that improves the quality of the goods firms supply. Our model develops a growth theory framework and we assume that firms spread around a circle. Our study assesses that spillovers between neighbors affect the probability of successful research for each of them. In particular, spillovers are the forces fuelling growth when, on the whole, firms turn out to be net receivers with respect to their neighbors.

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†Department of Economics, University of California Santa Cruz.
‡CODE, Universitat Autònoma de Barcelona.

1 Introduction

The distribution of economic activities across spaces experiences imbalances. A few areas display high concentrations of activities that entail the creation of economic wealth, while others are less endowed. This unequal distribution of resources is not a typical features of one specific region, but it is present, for instance, in the European countries as well as in the United States. The size of the spatial dimension does not matter a lot, since such inequalities may takes place inside a same country [1] or even across countries.[2] This phenomenon may have different origins. One can easily argue that it is mainly related to the distribution of natural resources. Firms (and in general economic activities) have a tendency to concentrate in regions exogenously endowed with natural resources, or in proximity to natural ways of communication and so on. These reasons have been the major forces driving the location of activities in the past during, for instance, the Industrial Revolution or, again, it may explains why Florida attracts so many (retired) people (Ottaviano-Thisse, 2003). This way of thinking reveal to be useless if we try to understand the rise of some concentrations less dependent from the natural advantages,as in the case of the Silicon Valley (Saxenian, 1994). Without denying the importance of the former elements, recently, economic literature has mainly addressed the attention to the second one. The purpose is to study the phenomenon of agglomeration as the result of human beings' actions . Economic geography has allowed to identify conditions such that regions attract activities, specialize production (see, for instance, the case of the Silicon Valley) and differ between *core and periphery.* By a *core and periphery* structure we mean an unequal distribution of activities that leads most or all of them to concentrate in a particular bounded area (the core) and leave the other part completely or nearly empty (periphery). Within a fairly balanced regional system, initial asymmetries are magnified by the cumulative causation and they produce a divergent process between regions. In those models the *cumulative causation* between the location of firms and workers yields self reinforcing movements explaining local specialization and affecting regional growth and decline. In this mechanism, increasing returns play a key role and they are mainly embodied in the fact that firms serve larger and larger markets. There are no linkages with the scarcity of local raw materials since production factors may be imported from somewhere else. Hence, the lack of natural resources is less crucial for the economic development (Krugman, 1991 and Krugman *et al.*, 1999).

The study of the linkages between local agglomerations and economic growth has been tackled from different viewpoints. Empirical evidence helps to explain to what extent local agglomerations support economic growth. The phenomenon of the industrial districts in Italy is the most striking case, but there are other interesting situations as that of the industrial reconversion in Wales

[1] Let us think of most of the European countries where northen regions are generally the richest regions in Belgium, Italy or Spain.

[2] Again European Union is an interesting case. If we focus on the per-capita level of income, northern countries record higher level - on average- than those in the South. In 2001 the average GDP per capita in EU was 23230 euro, ranging from 26640 euro in UK or 33200 euro in Danmark to, about, 11900 euro in Greece and Portugal (source New Cronos Database).

that has been planned thought the creation of local (more or less spontaneous) agglomerations of firms (Cooke and Morgan, 1998 and OECD, 1996). Nevertheless, even if from an empirical point of view it seems easy to verify the existence and the importance of local agglomerations as sources of growth, the same is not so evident from a theoretical viewpoint. The crucial element is to figure out the connection between growth and geography. Fujita and Thisse (2002) argue that one possible approach may consider agglomeration as the territorial counterpart of economic growth. In that sense, the process of development is similar to that of regional agglomerations. In order to study this dynamics they propose a model of endogenous growth for an economy with two regions. Their model relies on the combination of the building block of the Krugman's model core-periphery and the Grossman-Helpman-Romer's model of endogenous growth with horizontal differentiation. In particular, they introduce an R&D sector that uses skilled labor to produce new varieties for the modern sector. Focusing on the steady state equilibrium, they show that the growth of the economy is related to the spatial organization of the innovation sector across regions. In that sense, the R&D sector appears to be a strong centripetal force since it amplifies the circular causation of a core-periphery structure. In addition, they succeed also in proving that the growth effect is strong enough to make the level of welfare increase even in the peripheral areas. As in Grossman and Helpman (1991a and 1991b) growth is driven by the increase in the number of varieties. In particular, this result holds even if we concentrate on a case of local agglomerations, such as Marshallian districts. As argued by Basevi and Ottaviano (2002), in a district, the growth is related to the increase of the varieties of goods produced by firms. This last feature allows to distinguish between an approach that aims at concentrating on growth for a particular bounded space. Proximity matters and enhances the positive effects of investing in R&D. This last issue is not new. Usually, the idea of proximity entails the existence of local spillovers and looking at the microfoundations of local agglomeration (see Saxenian 1994), spillovers are also a centripetal (agglomerating) force. Empirical evidence, again, supports the idea that R&D proximity favors R&D intensity (see, for instance, Audretsch and Feldman, 1996, and Feldman and Audretsch, 1999), but, to our knowledge, none of the current researches aims at formalizing explicitly the role of local spillovers. Some example helps to explain this assessments. In order to point out the propensity of firms investing in R&D to cluser, an empirical study on Belgian data (Nicolini, 2002) figures out that this tendency appears in various sectors, as Figure 1 and 2 assess.

These two pictures clearly display that the distribution of firms investing in R&D in Belgium (in 1997) in these two sectors is far to be uniform. Some poles of concentration (or agglomeration) appear, even if the size of such concentrations may vary across sectors and across space. This tendency is manly explained by the idea that by clustering firms enjoy positive spillovers issuing from the other surrounding firms. These positive effects entail a reduction of the effort that each firm devotes to R&D activity as well as a reduction of the correspondent R&D costs each of them has to sustain. In addition to these pecuniar advantages, exploiting positive spillovers generated by the interaction with other firms (in the same sector, but even in other sectors whose activity is striclty connected with that we take into consideration) increases the likelihood of the success of R&D, i.e. R&D activity is more likely to lead to technological innovations. To quantify this effect,

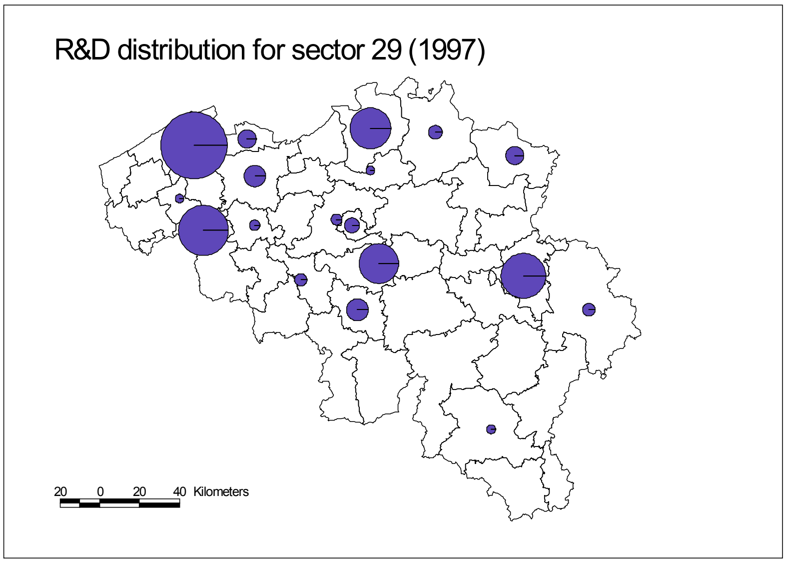

Figure 1: Distribution of Belgian firms investing in R&D. Sector 29: Manufacture of machines and equipment tools. (Source: Nicolini, 2002).

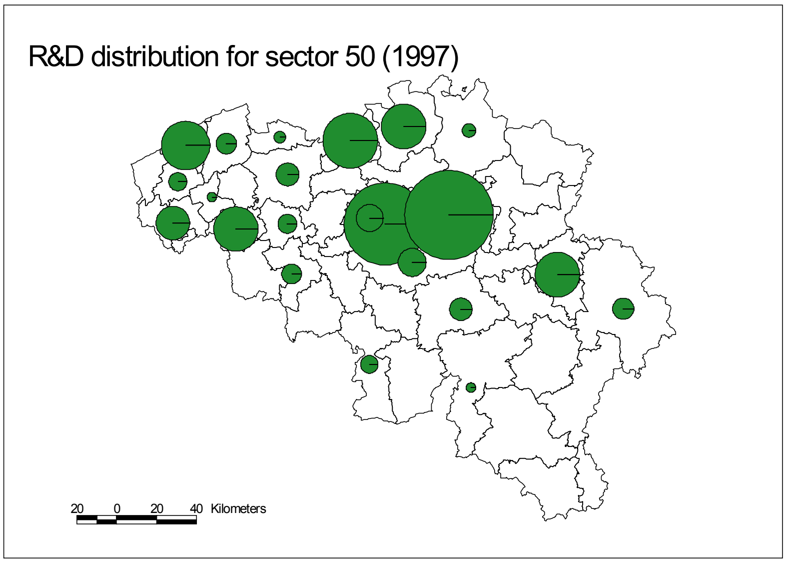

Figure 2: Distribution of Belgian firms investing in R&D. Sector 50: Commerce of means of transport (Source: Nicolini, 2002)

we report some data referring to Italian digital industrial districts[3] (RUR-CENSIS, 2001).

Table 1: Local firms and innovation (%) (Source: FEDERCOMIN-RUR/CENSIS)

	Industrial Districts	Local areas[4]
Firms'propensity to innovate	51	48.6
Share of profits invested in innovation projects in 1999	8.6	9.4

The previous table support the idea that when firms cluster and collaborate among them, they display a higher propensity to innovate and a lower financial effort.

As we stated above, the purpose of this paper is to concentrate on the formalization of the spillover effects in a growth model, assessing to what extent local spillovers can foster growth. To this end, we assume that knowledge produces a positive externality as in Lucas (1988). The technological change is due to intentional investment decisions as in Romer (1990), but, as a novelty, we add partial excludability to the positive technological externality. Spillovers will not effect the entire economy but only the locations closer to the source of technological advancements, i.e.the most immediate neighbors. This framework develops in the same direction as Grossman and Helpman (1991a and 1991b), Aghion and Howitt (1992) and Barro and Sala-i-Martin (1995). In particular, we grant to them the idea of a growth model with vertical quality ladder innovation.

In this study, we assume that clustes of innovating firms are due to local externalities that makes optimal for a firm to innovate intensely only if her neighbors do it. In this model, a final product is obtained by using different intermediate goods. Technological innovation, for each type of a fixed set of intermediate goods, takes the form of improvement occurring on a quality ladder. Innovations are generated by the private maximizing behavior of competitive research firms, or research labs, of the intermediate goods producers. For each intermediate product only the most innovative version of the good is provided, and the firm which patents the innovation enjoys monopoly profits until another firm develops a higher version of the good. Technology is a locally non excludable good, so that the probability with which a firm gets a successful innovation is proportional either to amount of research of the most immediate neighbors or to their relative innovation success.

Research intensity is on average constant across locations and innovation and growth arrive randomly among them. The economic forces operate in a way to keep growth on average similar across locations.

The behavior of each intermediate producer is such that the amount of research in any period depends upon the expected future research in its own sector and that of the two most immediate neighboring sectors in the present and in future periods. The expectation of higher future research in one of the intermediate sector discourages current research in the same sector because

[3] We intend as industrial district a concentration of firms belonging to a same sector of activity or other sectors vertically linked. Usually, firms firms belonging to it usually collaborate a lot among them or all of them participate to the development of common projects.

[4] By "local areas" we mean small and rising local forms of agglomeration that cannot be considered as industrial district yet.

it diminishes the expected duration of monopoly profits associated with the production of the most innovative intermediate good (the Schumpeterian creative destruction effect). Higher current neighbors research increases current research because it increases its current productivity. The expectation of higher future neighbor research, again, discourages current research because it makes future research more productive than the current one.

The remaining of the paper is organized as follows. In Section 2 we describe the building blocks of the model, then in Section 3 we analyze the correspondence between spillovers and growth and, finally Section 4 concludes.

2 The model

We develop our analysis in a setting similar to that proposed in Barro and Sala-i-Martin (1995) (granting also a lot to Grossman and Helpman (1991), and Aghion and Howitt (1992)). Nevertheless we differentiate in some assumptions. Our novelty is the introduction of a measure of local externalities between neighboring firms. In particular we intend to focus on the interaction between local externalities and economic growth.[5]

The economy is organized as follows: consumers (which dispose of one unit of labor that exchange for units of a final good), producers of a final good and producers of intermediate inputs.

2.1 The sector of final goods

In the economy there are N final goods. A final consumption good is produced under perfect competition, using m different intermediate inputs. These intermediate products appear in several varieties, and continuing improvements and refinements permanently increase the quality of the existing products. In each these sectors, the potential grades are arrayed along a quality ladder with rungs spaced at proportional intervals. In each sector j, the innovation of a new variety of an intermediate good replaces the old one and raises the quality q_{t_j} by a constant $q > 1$, so that in sector j at the t_j^{th} innovation

$$q_{t_j+1} = qq_{t_j} \tag{1}$$

where q_{0j} is the initial value of quality in sector j (taken as given). For sake of simplicity, we normalize it so that each good j begins with $q_{0j} = 1$. Subsequent improvements occur sequentially, jumping discretely one rung at a time, at the levels q, q_1, q_2, and so on. If in the j sector t_j improvements have already occurred, then the available grades in this sector are 1, q, q_1, q_2, ... , q_{t_j}.

[5] In this paper we assume that firms are exogenously located in the spirit of Salop (1979).

As in current literature (Barro and Sala-i-Martin, 1995) the production function for each firm i is assumed to be of a Cobb-Douglas type separable in terms of the intermediate inputs

$$y_i = A(L_i)^{1-\alpha} \sum_{j=1}^{m} (\widetilde{X}_{ij})^{\alpha} \text{ with } \alpha \leq 1,$$

where y_i represents the amount of final good produced by firm i, L_i the amount of unspecialized labor employed by final sector firm i, $\widetilde{X}_{ij} = \sum_{t=0}^{t_j} (q_t x_{i,j,t})$ is an index of the available intermediate inputs from sector j weighted by a function of their quality (q_t), and $x_{i,j,t}$ is the amount of intermediate good j of quality q_{t_j} used in firm i. The overall input in firm i from sector j, is therefore a quality weighted sum of the amount used of each grade. This additivity assumption implies that quality grades within a sector are perfect substitute as inputs in production. The way the production function is specified is such that the new quality offers an improvement in efficiency. Therefore, since the last innovation has an efficiency advantage over the prior innovations of the same sectors, and a disadvantage relative to the future ones, only the current best quality of each input will be used, and the production function simplifies to

$$y_i = A(L_i)^{1-\alpha} \sum_{j=1}^{m} (q_{t_j} x_{i,j})^{\alpha}. \tag{2}$$

We would like to point out that (2) displays constant return to scale keeping quality fixed, while increasing return to scale each time quality is improved.

2.2 The Intermediate producers

The producers of the m intermediate goods are assumed to be arranged on m locations along a circle. In each locations there is a very large number of firms who compete for been the innovation leader. We do not consider the location problem: we make the hypothesis that firms chose their location exogenously. While several firms can be active in research each period, only the one previously successful in innovation is allowed to produce.

The quality improvements modeled derives from the successful application of research efforts (namely labour), at the intermediate good level, in a non deterministic way.[6] Successful innovations arrive at random times.

For each sector, the interval $(t_j + 1) - t_j$ denotes the interval starting with the t_j^{th} innovation and ending just before the $(t_j + 1)^{th}$. During this interval, which has a different length for each values of t_j and for each intermediate sector, the best available quality is q_{t_j}. Hence, even if there

[6] As it will be specified in the next section, the utility is linear in consumption and, as in Aghion and Howitt (1992), there is no need to introduce a capital market for risk sharing.

are potentially many firms in the same intermediate sector j, only the one that successfully created the j^{th} intermediate good at quality level q_{t_j} retains the monopoly rights to produce it.

Let us suppose that each intermediate good is nondurable, and it is produced with unspecialized and specialized labor under constant return to scale, keeping quality constant. The total number of workers (that is also the total population) is $\overline{L}$. Each sector j hires L_j individuals supplying labor just for the manufacturing final good, while the intermediate sector hires workers that alternatevely for manufacturing of the intermediate product j or in research. For each period and in each intermediate sector, it holds that

$$l_j = l_j^x + l_j^{R\&D}.$$

Definition 1 *The technology of the production of each intermediate good takes a linear form.*

Hence, given that x_j represents both the number of unit produced and the amount of skilled labor allocated to manufacturing in sector j,we simplify the previous expression as follows

$$l_j = x_j + n_j, \tag{3}$$

where n_j denotes the amount of skilled labor allocated to R&D in sector j.[7]

2.3 Research and Development

The R&D success is random, so that progress occurs unevenly in the same sector and across sectors. We consider the number of persons hired in R&D, n_{j,t_j}, as the flow of resources used in R&D by all the potential innovators in sector j and the highest quality ladder reached in each sector j is t_j, with $j = 0, 1, 2, ..., m-1$.[8]

Definition 2 *Innovations in the sector j arrive randomly with the following arrival rate*

$$\delta_{j,t_j}{=}\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}})n_{j,t_j}, \textit{with } q_{t_{j-1}}, q_{t_j}, q_{t_{j+1}} > 1. \tag{4}$$

The probability of success in research results to be proportional to the amount of research in sector j, but also to some local characteristics. In this study, we consider the amount of the research effort of the two most immediate neighbor locations (henceforth neighbors) and a factor representing their technological progress with respect to the one in sector j. Let us define the term $\lambda(\cdot)$, representing the productivity of the research technology in sector j, as an increasing function of both the amount of research conducted by the two most immediate neighboring sectors $j-1$ and $j+1$, and their relative technological progress. Hence,

[7] For sake of semplicity, we do not account for a diversification in the qualification of the labor force (between manufacturing and reseach) that usually leads to a differentation of wages.

[8] Indirectly, this hypothesis refers to another one that will define later: the level of wages is normalized to one.

$$\frac{\partial \lambda(\cdot)}{\partial n_{j-1,t_{j-1}}} > 0, \qquad \frac{\partial \lambda(\cdot)}{\partial n_{j+1,t_{j+1}}} > 0, \qquad \frac{\partial \lambda(\cdot)}{\partial \frac{q_{t_{j-1}}}{q_{t_j}}} > 0, \qquad \frac{\partial \lambda(\cdot)}{\partial \frac{q_{t_{j+1}}}{q_{t_j}}} > 0.$$

This assumption embeds the idea of *local positive spillovers* that innovation exerts on neighbors. Being close to technologically advanced firms investing in R&D makes every unit spent in R&D more productive.

A priori we cannot understand if this positive externality generates an increase in the R&D in sector j, because of the increased productivity of research resources. Empirical literature seems confirming the tendency of innovating firms to cluster in order to exploit such positive externalities. Indeed the strength of local spillovers principally relies on physical proximity (see Audretsch and Feldman, 1996 as well as Feldman and Audretsch 1999). However, in this setting, the firm that succeeds in innovating monopolizes the intermediate sector, making previous period innovation obsolete.

2.4 Monopoly Price, Quantity and Profits

Taking into account the hypothesis of free entry into the R&D, each potential intermediate producer j has to find the optimal amount of labor resources to allocate to research. Each sector j is in equilibrium when the cost of making research equals the expected benefits. The cost of the research, per unit of time, is the flow of resources invested in R&D by the potential innovators in sector j. When the highest quality-ladder achieved in that sector is t_j, this cost is given by $w_{j,t_j} n_{j,t_j}$.

Definition 3 *Given the innovation flows, in each sector, only the highest quality good is produced and the producer enjoys a monopolist power.*

The benefit of the research occurs only when success arrives. This benefit is given by a stream of monopoly profits to the producer of the highest quality. The duration of this monopoly power is uncertain; it lasts until a competitor breaks through the next improvement. Not all the research is successful, therefore these expected returns arrive with the probability (4). In order to solve this problem, as in Barro- Sala-i-Martin (1995), we assume that the potential innovator cares only about the expected present value of the stream of profits, and not about the randomness of the returns of her research. We begin with computing the price, quantity and profit for the monopolist of the intermediate sector j. The producer of the intermediate good x_{j,t_j} using the t_j innovation (either discovered in her own lab or after purchasing the patent from another one) maximizes her profits from the sales of this intermediate good to all the final producers i. For sake of simplicity, we normalize the level of wages to one.

Since the final good y_i is produced under perfect competition, the price of the intermediate good that enters as input into the production function of y_i has to equalize the marginal product, which in our specification of Cobb Douglas production function becomes

$$p_{i,j,t_j}(x_{i,j,t_j}) = \frac{\partial y_i}{\partial x_{i,j,t_j}} = A L_i^{1-\alpha} \alpha q_{t_j}^{\alpha} x_{i,j,t_j}^{\alpha-1}, \tag{5}$$

Knowing that only the highest quality of each good is purchased, by aggregating the profit maximizing conditions over all final good producers we get the demand function for good x_{j,t_j}

$$x_{j,t_j} = L \left[\frac{A\alpha q^{t_j}}{p_{j,t_j}} \right]^{\frac{1}{1-\alpha}}. \tag{6}$$

Replacing (5) and (6) into the profits of the leading hedge producer j, the interior solution for $x_{j,t_j} > 0$ of the first order conditions (second order always satisfied) yields the monopoly price

$$p_{j,t} = \frac{1}{\alpha}. \tag{7}$$

Plugging (5) into (6), the aggregate quantity produced of the j^{th} intermediate good of leading edge quality becomes

$$x_{j,t_j} = L \left[A\alpha^2 q_{t_j}^{\alpha} \right]^{\frac{1}{1-\alpha}}, \tag{8}$$

implying that an evolution of t_j over time in each sector and its divergence across sectors leads to variations of the quality over time and across sectors. Combining (5) with (6) and taking into account the costs of production, the profits for a temporal monopolist turns out to be:

$$\pi_{j,t_j} = \left(\frac{1-\alpha}{\alpha} \right) x_{j,t_j} = \left(\frac{1-\alpha}{\alpha} \right) L \left[A\alpha^2 q_{t_j}^{\alpha} \right]^{\frac{1}{1-\alpha}}. \tag{9}$$

2.5 The expected value of the research

The leader in sector j, producing the intermediate good at quality q_{t_j}, enjoys monopoly profits from the time she made the discovery τ_{t_j} until the time a competitor will come up with the next innovation q_{t_j+1} at time τ_{t_j+1}. Let Γ_{j,t_j} denote the time interval over which t_j retains the leadership

$$\Gamma_{j,t_j} = \tau_{t_j+1} - \tau_{t_j}.$$

Let us call V_{j,t_j} the present value of this stream of profits. Assuming r constant

$$V_{j,t_j} = \int_0^{\Gamma_{j,t_j}} \pi_{j,t_j} e^{-rs} ds.$$

Over the period Γ_{j,t_j} the profits are constant: both q_{t_j} and w_{j,t_j} do not vary. So the solution for the benefit from the t_j innovation entails that this benefit increases with the amount of profit and their duration

$$V_{j,t_j} = \frac{\pi_{j,t_j}(1 - e^{-r\Gamma_{j,t_j}})}{r}. \tag{10}$$

However, the duration Γ_{j,t_j} is a random variable, so that the value of interest for the intermediate producer is actually the expected benefit $E(V_{j,t_j})$. In order to determine this value, we need to fix the probability per unit of time that innovation occurs at a point in time τ. Let us define this probability $g(\tau)$. This density function can be found by computing the derivative of the cumulative probability density function $G(\tau)$ for Γ_{j,t_j}. Then, $G(\tau)$ describes the probability that $\Gamma_{j,t_j} \leq \tau$. The probability per unit of time that innovation occurs at τ is equal to the probability of one occurring per unit of time δ_{j,t_j} (equation (4)) conditional on a discovery not having happened so far $[1 - G(\tau)]$

$$g(\tau) = \frac{dG(\tau)}{d\tau} = [1 - G(\tau)]\, \delta_{j,t_j}. \tag{11}$$

Throughout Γ_{j,t_j}, n_{j,t_j} the number of workers in R&D is constant, but we are not able to argue the same for all the components of (4). Innovations takes place at different times in different sectors, making R&D workforce react as well. If a steady state exists, then the optimal research intensity needs to be constant at the optimal level, and the technological difference between sectors would, on average, be constant as well. In the next section we prove that an equilibrium exists, so that we can treat the independent variables in (4) as constant (on average) during Γ_{j,t_j}. Therefore, for the Jensen's Inequality, the solution that we obtain when evaluating δ_{j,t_j} as a constant, and not constant on average, is a limit solution instead of an exact solution. Therefore, by considering δ_{j,t_j} as a constant, we solve the differential equation.

$$\dot{G}(\tau) + p_{j,t_j} G(\tau) = \delta_{j,t_j},$$

and we define a solution for the density function

$$G(\tau) = 1 - e^{-\delta_{j,t_j}\tau},$$

such that

$$g(\tau) = \dot{G}(\tau) = \delta_{j,t_j} e^{-\delta_{j,t_j}\tau}.$$

This last result allow to compute the expected benefit of the research

$$\begin{aligned} E(V_{j,t_j}) &= E\left(\frac{\pi_{j,t_j}(1 - e^{-r\Gamma_{j,t_j}})}{r}\right) = \int_0^\infty \frac{\pi_{j,t_j}(1 - e^{-r\tau})}{r} \delta_{j,t_j} e^{-p_{j,t_j}\tau} d\tau \\ &= \frac{\pi_{j,t_j}}{r + \delta_{j,t_j}}. \end{aligned} \tag{12}$$

Plugging (9) into (12) one gets the full expression for the expected value of successful research. Anyway, a few comments may be drawn on (12). The denominator of this expression displays the obsolescence adjusted interest rate, namely $r + \delta_{j,t_j}$, where the term δ_{j,t_j} represents the effect of creative destruction. When the amount of research conducted by the neighbors and their relative technological advancements is constant, a high level of research in sector j diminishes the expected payoffs from innovation in sector j. Generally, a highly innovative sector displays higher probability of obsolescence too. This mechanism leads the monopoly profit to be enjoyed for a shorter period of time. Similarly, considering the level of research in sector j as a constant, high contemporaneous level of research of neighbors increases the probability of successful research in sector j. As before, this process shortens the length of the monopoly profit for the incumbent producer in sector j, and, as a consequence of that, decreases the expected value of the research, namely the payoff from innovation.

2.6 The determination of the R&D effort

Now, we need to figure out more precisely the trade-off between costs and benefits in doing R&D. As we explained above, the uncertainty related to the R&D outcomes is proportional to the probability that an innovation occurs. We already know that it arrives randomly at the already mentioned rate δ_{j,t_j}.

In order to be the monopolist of invention $t_j + 1$, the cost of research has to be advanced during innovation period t_j, while the benefits, if ever, are going to be enjoyed during invention period $t_j + 1$.

The costs of research per unit of time during the invention period t_j are simply the amount of money paid to the employees in the research sector. Taking into account the nomalization of wages they are equal to n_{j,t_j}. Similarly, the benefits of an innovations may be defined as the product between the probability δ_{j,t_j} per unit of time of success, and with it the enjoyment of $E(V_{j,t_j})$ during invention period $t_j + 1$ (i.e. $\delta_{j,t_j} E(V_{j,t_j+1})$).

In equilibrium, with $n_{j,t_j} \geq 0$, for each potential innovator the cost of an additional unit of research should be equalized to the benefit $\delta_{j,t_j} E(V_{j,t_j+1})$, i.e. the value of becoming the $t_j + 1$ monopolist multiplied by the probability of success in innovating of an extra research unit

$$\lambda(\frac{q^{t_{j-1}}}{q^{t_j}}, n_{j-1,t_{j-1}}, \frac{q^{t_{j+1}}}{q^{t_j}}, n_{j+1,t_{j+1}})E(V_{j,t_j+1}) \leq 1, \qquad n_{j,t_j} \geq 0. \tag{13}$$

High levels of contemporaneous research in more advanced neighborhood $j-1$ and $j+1$ increases the productivity of research, i.e. the probability of success, in sector j. This change determines an increase in the value of research, which leads to higher level of research in sector j.

Replacing (4) and (12) into (13), the equilibrium condition becomes

$$\frac{1}{\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}})} \geq \frac{\pi_{j,t_j+1}}{r + \lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}})n_{j,t_j+1}},$$

with at least one equality holding. Neighbors' current research decreases the cost of current research. As it has been argued, the positive spillovers are due to the increase in productivity of one sector's research once neighbors are researching intensively. Neighbors future research, instead, decreases the benefit for the sector current research. The cause of this negative effect is that an expected increase in productivity of research increases one's sector current research obsolete rate, and shortening the period for which monopolistic profit could be enjoyed it decreases the expected benefit of research.

Manipulating the previous condition by replacing the profit function at period invention t_j+1 yields to

$$r+\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}+1},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}+1})n_{j,t_j+1}=\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}})\left(\frac{1-\alpha}{\alpha}\right)L\left[A\alpha^2 q^{\alpha}_{t_j+1}\right]^{\frac{1}{1-\alpha}},$$

and applying (1) we get

$$\frac{1}{q^{\alpha}_{t_j}}=\frac{\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}})\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}}LA^{\frac{1}{1-\alpha}}q^{\alpha}\left[q^{\alpha}_{t_j+1}\right]^{\frac{-\alpha}{1-\alpha}}}{r+\lambda(\frac{q_{t_{j-1}}}{q_{t_j}},n_{j-1,t_{j-1}+1},\frac{q_{t_{j+1}}}{q_{t_j}},n_{j+1,t_{j+1}+1})n_{j,t_j+1}}. \tag{14}$$

At each point in time, the labor market condition has to hold, so that

$$l_j-n_{j,t_j}=L\left(A\alpha^2\right)^{\frac{1}{1-\alpha}}\left[q^{\alpha}_{t_j}\right]^{\frac{\alpha}{1-\alpha}}$$

or, put differently,

$$\frac{1}{q^{\alpha}_{t_j}}=L^{(1-\alpha)}A\alpha^2\left(l_j-n_{j,t_j}\right)^{-(1-\alpha)}. \tag{15}$$

Since the amount of research will ultimately depend on neighbors' research and technological progress, the wage will vary depending on location too.

In the next section we move directly to the results, while technical details to get them are presented in the Appendix.

3 Spillover effects and growth

At the steady state, the values of (n_{j,t_j},q_{t_j}), correspond to (knowing that $q=1$),

$$\overline{n_j} = \frac{(1-\alpha)}{2\alpha - 1}\left[l_j - \left(\frac{r}{\left(\frac{1-\alpha}{\alpha}\right)Q}\right)^{\frac{1}{1-c-d}}\right], \tag{16}$$

$$\overline{q}_{t_j} = \left\{L^{(1-\alpha)}A\alpha^2\left[l_j - \frac{(1-\alpha)}{2\alpha-1}\left[l_j - \left(\frac{r}{\left(\frac{1-\alpha}{\alpha}\right)Q}\right)^{\frac{1}{1-c-d}}\right]\right]^{-(1-\alpha)}\right\}^{\frac{1}{\alpha}}. \tag{17}$$

In particular, recalling that $Q \equiv \frac{q_{t_{j+1}}}{q_{t_j}}\frac{q_{t_{j+1}}}{q_{t_j}}$ in any period, we may draw some interesting considerations. For any value of $Q > 0$, at time t, a positive variation of it implies that the quality of the neighbors betters with respect to that of t_j.[9] Given the value of all other parameters, an increase of the value of Q entails an increase of the number of qualified workers working in R&D in sector t_j ,[10] that according to (17) yields to an improvement of the level of quality q_{t_j} . The change in q_{t_j} is due to the positive spillover effects that sector t_j receives from (t_{j-1}) and (t_{j+1}) and its own expenses in R&D that make an innovation in t_j more likely to happen. In addition, if we concentrate on the size of the sector we take into account (i.e the size of the parameter l_j),we deduce that an increase of l_j entails an increase of the number of workers devoted to R&D $(\overline{n_j})$. Nevertheless, for low values of α(in particular for $\alpha \leq 2/3$) the increase of the number of qualified workers is lower than the increase of number of workers l_j and quality $\overline{q}_{t_j}$will definitely records positive variations. In case $\alpha \geq 2/3$, the variation of $\overline{q}_{t_j}$ with respect to an increase of l_j is uncertain.

Conclusion 4 *A big size of local spillovers entails an increase of the number of workers in the R&D sector as well as the level of the highest quality available in sector j. Similarly, a big size of the sector j (via the parameter l_j) yields to a high number of workers involved in R&D and for low values of α the level of quality will be definitely better.*

In this model, the solution of the dynamics system allow to argue that, the main forces driving changes in the setting are spillovers. In particular, we are interested in determining to what extent spillovers may drive economic growth. In order to state this effect, we need to define the expression for the aggregate output and the total quantity of intermediaries produced as function of local spillovers. By aggregating (8) across sectors, we get :

[9] In case of negative variation of Q, namely, the quality q_{t_j} betters in comparison to that of its neigbors, the result we get are symmetrically the opposite, because the neigbours try to fill the existing gap and this makes sector t_j loose its leading position.

[10] Firms in t_j realises that they have to invest more in R&D for reducing the gap if they aim at achieveng a monipolistic power.

$$X = A^{1/(1-\alpha)}\alpha^{\frac{2}{1-\alpha}}L\sum_{j=1}^{N}(q_{t_j})^{\frac{\alpha}{1-\alpha}} \tag{18}$$
$$= A^{1/(1-\alpha)}\alpha^{\frac{2}{1-\alpha}}L\overline{Q}$$

Plugging (8) into (2) and aggregating over the firms i, we obtain the expression for the aggregate output:

$$Y = A^{1/(1-\alpha)}\alpha^{\frac{2\alpha}{1-\alpha}}L\sum_{j=1}^{N}(q_{t_j})^{\frac{\alpha}{1-\alpha}}. \tag{19}$$

At the steady state, it holds that for any firm i, q_{t_j} is equal to (17), so that the quantity relies on the value of the spillover parameter Q. Indeed both Y and X rely on the index $\overline{Q} = \sum_{j=1}^{N}(q_{t_j})^{\frac{\alpha}{1-\alpha}}$. At the equilibrium, by some algebraic manipulation of (8), it is easily to fix that $(q_{t_j})^{\frac{\alpha}{1-\alpha}} = LA^{1/(1-\alpha)}\alpha^{\frac{2}{1-\alpha}}(\overline{x}_j)$,where $\overline{x}_j = (l_j - \overline{n_j})$.

Local spillovers may be a source of growth, but, since they can be positive or negative according to the size of $\left(q_{t_{j+1}}, q_{t_j}, q_{t_{j+1}}\right)$, we need to pay attention for this detail when defining the growth conditions. In particular, we concentrate on the case when $\alpha = 2/3$ and $c = d = 0$. Under those hypothesis, according to (16) and (17), it is possible to state that :[11]

$$\gamma_Y = \gamma_X = -\gamma_Q.$$

Under those hypothesis, $\overline{n_j}$ reduces such that :

$$\overline{x}_j = \frac{2r}{Q}.$$

Hence, both Y and X are constant multiples of Q.[12] In order to quantify how spillovers affect growth, we need to define the rate of growth at the steady state. By applying logarithm and derivatives to Q, and exploiting (1), we obtain that, at the steady state :

[11] If we took into account that $c + d < 1$, we would get $\gamma_x = -\frac{1}{1-c-d}\gamma_Q$,i.e. the relationship between the two rates of growth would be magnified by a positive constant.

[12] We remind that the interest rate r is taken as constant. Moreover, at the steady state, the difference of qualities between sectors are constant, so the aggregate index is the product between the number of sectors (less two) and the costant level of spillovers for each couple.

$$\gamma_Q = \left[\frac{1}{q_{t_{j+1}}} + \frac{1}{q_{t_{j-1}}} - \frac{2}{q_{t_j}}\right]. \tag{20}$$

Looking at expression (20), we may easily state that in case any sector t_j is a net receiver of spillovers from its two neighbors,[13] its quality improves and $\gamma_Q < 0$. Now, assuming that the level of quality for each sector at time t is greater than one, by computing the rate of growth of x (via expression 3), we obtain that γ_x is positive for $\gamma_Q < 0$. Hence, once a sector benefits from positive spillovers, γ_x is expected to be positive as well as γ_Y (according to equation (2)).On the contrary, whenever $\gamma_Q > 0$, any sector t_j is a net source of spillovers, and both γ_Q and γ_X will suffer from it.

Conclusion 5 *At the steady state, for $\alpha = 2/3$, when each sector enjoys positive spillovers from his neigbours, the rate of growth of the system is positive too.*

3.1 Households spending and welfare analysis

Up to now, we did not explicitly mention a particular form of utility function of consumers. Indeed, we implicity consider that in this closed economy all the total quantity of final good (Y) is consumed by the all the population ($\overline{L}$). In order to be more precise, as in Barro and Sala-i Martin (1995), we assume a constant intertemporal elasticity function (CIES). We consider that households maximize the utility function

$$U = \int_0^{\infty} \left(\frac{c^{1-\theta} - 1}{1-\theta}\right) e^{-\rho t} dt, \tag{21}$$

where c represents consumption, ρ is the rate of intertemporal preference and the elasticity of the marginal utility is (-θ). Household aggregate income is the sum of wages on the fixed aggregate quantity of labor and interest on the total assets (market values of all the firms). Household optimization gives the standard condition for consumption growth rate

$$\gamma_c = \frac{1}{\theta}(r - \rho). \tag{22}$$

In order to compute the steady state conditions, in the model, we assumed that the interest rate is given as constant. We proceed to check this hypothesis, once we have computed the rate of growth. Knowing that, in our setting, $Y = C$, at the steady state, we may derive that $\gamma_c = \gamma_Q$. By comparing (20) with (22), we get that

[13] This conclusion is involved by the assumption that t_j, t_{j-1}, and t_{j+1} represent the latest quality rung in sector j, $j-1$, and $j-1$ respectively, at the same point in time they are all different. Or, put differently, the same quality steps are reached by the sectors at different points in time.

$$r = \theta \left[\frac{1}{q_{t_{j+1}}} + \frac{1}{q_{t_{j+1}}} - \frac{2}{q_{t_j}} \right],$$

that is constant for the given values of the parameter $(q_{t_j}, q_{t_{j+1}}, q_{t_{j+1}})$ at the equilibrium.

In particular, knowing that Y is the real output (namely the flow of consumption goods) in the economy, we aim at assessing some considerations considering the level of welfare when local spillovers are present.

At the steady state, we begin with taking into account equation (16). From that equation is easly to compute that :

$$\overline{x}_j = \frac{3\alpha - 2}{2\alpha - 1} l_j + \frac{1 - \alpha}{2\alpha - 1} \left[\frac{r}{\frac{1-\alpha}{\alpha} Q} \right]^{\frac{1}{1-c-d}}. \tag{23}$$

Knowing that expession, via $(q_{t_j})^{\frac{\alpha}{1-\alpha}} = LA^{1/(1-\alpha)} \alpha^{\frac{2}{1-\alpha}} (\overline{x}_j)$,we comupte the value of (q_{t_j}). Plugging it into (19) and aggregating across sectors, we obtain the following expression:

$$Y = A^{2/(1-\alpha)} \alpha^{\frac{2(\alpha+1)}{1-\alpha}} L \left[\frac{3\alpha - 2}{2\alpha - 1} (Nl_j) + \frac{1 - \alpha}{2\alpha - 1} \left[\frac{Nr}{\frac{1-\alpha}{\alpha} NQ} \right]^{\frac{1}{1-c-d}} \right]. \tag{24}$$

Looking at (24), we realize that positive variations of Y correspond to negative variations of Q. Thinking of Y as the flow of the consumption goods, its values will be high as far as the level of Q is small. Put differently, in economic terms, we may assess that the level of welfare of the system increases when the disparities across sectors are small, namely when the size of the spillovers is not too big.

Conclusion 6 *Consumers enjoy better low levels of spillovers .*

At this point, a clear trade off on the nature and effects of local spillover appears. On one hand, positive spillovers allow to better the local quality of goods. On the other hand, a high level of positive spillovers entail high discrepancies across intermediate sectors that produce a reduction of the level of welfare of the local population (intended as the flow of consumption at the steady state). Therefore, supporting policies that incite collaborations among firms such to reduce the quality gap among them is definitely welcome in order to improve the economic welfare of the system. Nevertheless, reducing such a gap implies also smoothing the magnitude of the rate of growth at the equilibrium.

4 Conclusions

In this study we assess the role of local spillovers as engine of economic growth. We formalize the existence of positive spillovers between neighbors by assuming that sectors (and firms) settle around a circle. Each of them interacts with the two sectors next to it. In particular, as in other growth models, positive spillovers concern the R&D activity and the likelihood to create an innovation when locating next to other firms that invest in R&D.

Nevertheless, spillovers may have a double effects. On one hand, considering the level of research of a sector as a constant, a high level of research of neighbors increases the probability of successful research but also the osbsolescence rate. On the other hand, neighbor future research decreases the benefits for the current research of a given sector. Moving to the analysis of spillovers, spillover growth rate drives the growth rate of the economy for fixed values of the parameters. Under those hypothesis we are able to assert that when sectors (as a whole) are net receivers of spillovers (namely, we are in a situation of positive spillovers) spillovers drive the economic growth. The opposite happens when, on the whole, sectors are net suppliers of spillovers.

In that sense, if we would extend these results to a wider context to provide suggestions in political matters, it could be reasonable to think that policy agents can profitable sustain the creation of agglomerations (or networks) as a way of fostering local growth under the conditions of preventing the creation of leading positions among firms belonging to the agglomeration. Hence, policies that sustain a sharing knowledge process across the members of a group are useful to guarantee the active role of local agglomerations in economic systems. However, focusing on the welfare analysis, we may also add that high discrepancies among the quality of goods supplied by firms entail also a negative effect on the local welfare, since a high level of spillovers yields proportionally to a low level of consumption.

More developments in this direction should deserve attention. In particular, an analysis on the effect of the welfare could be useful to understand to what extent local agglomerations can help the local development. In addition, another possible extensions could involve the fomalization of spillover effects in an economic setting involving (explicitly) the creation of local agglomerations, where the location of firms is not exogenously given.

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A Appendix

Before describing mathematical computations, it is important to remind that t_j is not a particular instant in time, but the latest quality step in sector j, so that for different sectors the t_j, t_{j-1}, t_{j+1} will signify different numbers. Combining (14) and (15), we get

$$L^{(1-\alpha)} A\alpha^2 \left(l_j - n_{j,t}\right)^{-(1-\alpha)} =$$

$$= \frac{\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}}) \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} LA^{\frac{1}{1-\alpha}} q^{\alpha} \left[L^{(1-\alpha)} A\alpha^2 \left(l_j - n_{j,t_j+1}\right)^{-(1-\alpha)}\right]^{\frac{-\alpha}{1-\alpha}}}{r + \lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}+1}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}+1}) n_{j,t_j+1}},$$

that can be reduced to

$$\frac{1}{\left(l_j - n_{j,t_j}\right)^{(1-\alpha)}} = \frac{\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}}) \left(\frac{1-\alpha}{\alpha}\right) q^{\alpha} \left(l_j - n_{j,t_j+1}\right)^{\alpha}}{r + \lambda(\frac{q^{t_{j-1}+1}}{q^{t_j+1}}, n_{j-1,t_{j-1}+1}, \frac{q^{t_{j+1}+1}}{q^{t_j+1}}, n_{j+1,t_{j+1}+1}) n_{j,t_j+1}}. \quad (25)$$

Again, an anticipated increase in the research efforts in one sector discourages current research (*creative destruction effect*), shortening the expected lifetime of innovation monopoly.

Let us define $\phi \equiv \left(\frac{1-\alpha}{\alpha}\right) q^{\alpha}$, (25) becomes

$$r + \lambda(\frac{q^{t_{j-1}+1}}{q^{t_j+1}}, n_{j-1,t_{j-1}+1}, \frac{q^{t_{j+1}+1}}{q^{t_j+1}}, n_{j+1,t_{j+1}+1}) n_{j,t_j+1} =$$

$$= \phi\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}}) \left(l_j - n_{j,t+1}\right)^{\alpha} \left(l_j - n_{j,t}\right)^{(1-\alpha)}.$$

Let us remind that t_j, t_{j-1}, and t_{j+1} represent the latest quality rung in sector j, $j-1$, and $j-1$ respectively, at the same point in time they are all different, i.e. the same quality steps are reached by the sectors at different point in time.

In addition, we select a specific form for the expression (4) such that

$$\lambda(\frac{q_{t_{j-1}}}{q_{t_j}}, n_{j-1,t_{j-1}}, \frac{q_{t_{j+1}}}{q_{t_j}}, n_{j+1,t_{j+1}}) \equiv \left(\frac{q_{t_{j-1}}}{q_{t_j}} \frac{q_{t_{j+1}}}{q_{t_j}}\right) \vartheta(n_{j-1,t_{j-1}}, n_{j+1,t_{j+1}}),$$

and

$$\lambda(\frac{q^{t_{j-1}+1}}{q^{t_j+1}}, n_{j-1,t_{j-1}+1}, \frac{q^{t_{j+1}+1}}{q^{t_j+1}}, n_{j+1,t_{j+1}+1}) \equiv \left(\frac{q^{t_{j-1}+1}}{q^{t_j+1}}\frac{q^{t_{j+1}+1}}{q^{t_j+1}}\right)\vartheta(n_{j-1,t_{j-1}+1}, n_{j+1,t_{j+1}+1}).$$

Since what it is really important for sector j is t, we also define $\vartheta(\tau) \equiv \vartheta(n_{j-1,t_{j-1}}, n_{j+1,t_{j+1}})$, and $\vartheta(\tau+1) \equiv \vartheta(n_{j-1,t_{j-1}+1}, n_{j+1,t_{j+1}+1})$, since we are interested in determining to what extent the amount of contemporaneous neighbors' research is really important for sector j . Hence, the equilibrium condition becomes

$$\left(l_j - n_{j,t_j}\right) = \left[\frac{r + \left(\frac{q^{t_{j-1}+1}}{q^{t_j+1}}\frac{q^{t_{j+1}+1}}{q^{t_j+1}}\right)\vartheta(\tau+1)n_{j,t_j+1}}{\phi\left(\frac{q_{t_{j-1}}}{q_{t_j}}\frac{q_{t_{j+1}}}{q_{t_j}}\right)\vartheta(\tau)\left(l_j - n_{j,t_j+1}\right)^{\alpha}}\right]^{\frac{1}{(1-\alpha)}},$$

whose dynamics we study after having introduce this substitution $x_t = \left(l_j - n_{j,t_j}\right)$

$$x_t = \left[\frac{r + \left(\frac{q^{t_{j-1}+1}}{q^{t_j+1}}\frac{q^{t_{j+1}+1}}{q^{t_j+1}}\right)\vartheta(\tau+1)(l_j - x_{t+1})}{\phi\left(\frac{q_{t_{j-1}}}{q_{t_j}}\frac{q_{t_{j+1}}}{q_{t_j}}\right)\vartheta(\tau)x_{t+1}^{\alpha}}\right]^{\frac{1}{(1-\alpha)}}.$$

It can be easily checked that when neighbors research intensity does not vary, $\vartheta(\tau) = \vartheta(\tau+1)$, sector j research intensity present a unique crossing with the steady state line in the set of existence of the function.

We can conclude that a steady state exists for which $x_t = x_{t+1}$. This implies that for all sectors j, $j = 0, 1, 2...m-1$, it holds that $n_{j,t_j} = n_{j,t_j+1} = \overline{n_j}$, which implies $\vartheta(\tau) = \vartheta(\tau+1) = \vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})$, while q_{t_j}, $q_{t_{j-1}}$ ... at each point in time are considered exogenously given in each sector.

The steady state equation becomes

$$r + \left(\frac{q_{t_{j-1}+1}}{q_{t_j+1}}\frac{q_{t_{j+1}+1}}{q_{t_j+1}}\right)\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})\overline{n_j} = \phi\left(\frac{q_{t_{j-1}}}{q_{t_j}}\frac{q_{t_{j+1}}}{q_{t_j}}\right)\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})\left(l_j - \overline{n_j}\right).$$

We define $Q \equiv \frac{q_{t_{j-1}}}{q_{t_j}}\frac{q_{t_{j+1}}}{q_{t_j}}$, and via expression (1) we deduce even that $Q = \frac{q_{t_{j-1}+1}}{q_{t_j+1}}\frac{q_{t_{j+1}+1}}{q_{t_j+1}}$, and we get

$$\overline{n_j} = \frac{\phi l_j \vartheta(\overline{n}_{j-1}, \overline{n}_{j+1}) - rQ^{-1}}{(1+\phi)\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})}.$$

In order to find a finite solution for $\overline{n_j}$, $\overline{n}_{j-1}$, and $\overline{n}_{j+1}$ we need to specify the function $\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})$ and solve for the difference equation

$$\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1})\left[l_j - \frac{(1-\phi)}{\phi}\overline{n_j}\right] = \frac{r}{\phi}Q^{-1}$$

The solution method requires a linearization of the expression, so we can assume a the following (convenient) functional form for $\vartheta(\cdot)$.

$$\vartheta(\overline{n}_{j-1}, \overline{n}_{j+1}) = \left[l_{j-1} - \frac{(1-\phi)}{\phi}\overline{n}_{j-1}\right]^{-d} \left[l_{j+1} - \frac{(1-\phi)}{\phi}\overline{n}_{j+1}\right]^{-c},$$

with $c, d \geq 0$ such to satisfy the assumption that an increase in neighbors research intensity generate a positive technological spillover. Taking logs and defining $z_j \equiv \log\left[l_j - \frac{(1-\phi)}{\phi}\overline{n_j}\right]$ and $\xi \equiv \log\left(\frac{r}{\phi}Q^{-1}\right)$, the second order difference equation becomes:

$$-cz_{j+2} + z_{j+1} - dz_j = \xi.$$

The particular integral is

$$z_p = \frac{\xi}{1-c-d},$$

and since ξ is positive, in order to z_p to be positive we need to impose $c + d < 1$.

The characteristic equation presents the roots: $b_{1,2} = \frac{1}{2c} \pm \frac{1}{2c}\sqrt{1-4cd}$. In order to have two distinct real roots, c and d have to satisfy $1 - 4cd \geq 0$ or $cd \leq \frac{1}{4}$. Since the previous restriction was $c + d < 1$, this second condition will be always satisfied, and the general solution becomes

$$z_j = \frac{\xi}{1-c-d} + A_1\left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^j + A_2\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^j.$$

Imposing the initial condition $z_0 = \frac{\xi}{1-c-d} + A_1\left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^0 + A_2\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^0$we find

$$A_1 = z_0 - \frac{\xi}{1-c-d} - A_2.$$

Imposing periodicity (i.e. if we have m sectors arrayed along a circle it must be that the first one coincides with the one after the last one, so if z_0 is the first one z_{m-1} is the last one, and the condition becomes $z_0 = z_m$)

$$\frac{\xi}{1-c-d} + A_1 + A_2 = \frac{\xi}{1-c-d} + A_1\left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m + A_2\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m,$$

and from this, after substituting the previously found A_1, we obtain

$$A_1 = \left(z_0 - \frac{\xi}{1-c-d}\right)\left[\frac{-1 + \left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m}{\left(\frac{1}{2c} + \frac{1}{2c}\sqrt{1-4cd}\right)^m - \left(\frac{1}{2c} - \frac{1}{2c}\sqrt{1-4cd}\right)^m}\right],$$

and

$$A_2=\left(z_0-\frac{\xi}{1-c-d}\right)\left[\frac{1-\left(\frac{1}{2c}-\frac{1}{2c}\sqrt{1-4cd}\right)^m}{\left(\frac{1}{2c}+\frac{1}{2c}\sqrt{1-4cd}\right)^m-\left(\frac{1}{2c}-\frac{1}{2c}\sqrt{1-4cd}\right)^m}\right],$$

so that the solution becomes

$$\begin{aligned} z_j &= \frac{\xi}{1-c-d}+\left(z_0-\frac{\xi}{1-c-d}\right)\left[\frac{-1+\left(\frac{1}{2c}+\frac{1}{2c}\sqrt{1-4cd}\right)^m}{\left(\frac{1}{2c}+\frac{1}{2c}\sqrt{1-4cd}\right)^m-\left(\frac{1}{2c}-\frac{1}{2c}\sqrt{1-4cd}\right)^m}\right]\cdot \\ &\cdot\left(\frac{1}{2c}-\frac{1}{2c}\sqrt{1-4cd}\right)^j+\left(z_0-\frac{\xi}{1-c-d}\right)\cdot \\ &\cdot\left[\frac{1-\left(\frac{1}{2c}-\frac{1}{2c}\sqrt{1-4cd}\right)^m}{\left(\frac{1}{2c}+\frac{1}{2c}\sqrt{1-4cd}\right)^m-\left(\frac{1}{2c}-\frac{1}{2c}\sqrt{1-4cd}\right)^m}\right]\left(\frac{1}{2c}+\frac{1}{2c}\sqrt{1-4cd}\right)^j. \end{aligned}$$

Solving for z_m, we obtain $z_m=\frac{\xi}{1-c-d}$, so that it must be true that $z_0=\frac{\xi}{1-c-d}$, which implies for our solution that

$$z_j=\frac{\xi}{1-c-d}\qquad \forall j=0,1,2,...,m-1.$$

Recalling $z_j\equiv\log\left[l_j-\frac{(1-\phi)}{\phi}\overline{n_j}\right]$ and $\xi\equiv\log\left(\frac{r}{\phi}Q^{-1}\right)$, the solution for the optimal value of research intensity is

$$\overline{n_j}=\frac{\phi}{(1-\phi)}\left[l_j-\left(\frac{r}{\phi}Q^{-1}\right)^{\frac{1}{1-c-d}}\right],$$

and since $\phi\equiv\left(\frac{1-\alpha}{\alpha}\right)q^\alpha$, the solution becomes:

$$\overline{n_j}=\frac{(1-\alpha)q^\alpha}{\alpha-(1-\alpha)q^\alpha}\left[l_j-\left(\frac{r}{\left(\frac{1-\alpha}{\alpha}\right)q^\alpha Q}\right)^{\frac{1}{1-c-d}}\right],$$

$$\overline{q}_{t_j}=\left\{L^{(1-\alpha)}A\alpha^2\left[l_j-\frac{(1-\alpha)q^\alpha}{\alpha-(1-\alpha)q^\alpha}\left[l_j-\left(\frac{r}{\left(\frac{1-\alpha}{\alpha}\right)q^\alpha Q}\right)^{\frac{1}{1-c-d}}\right]\right]^{-(1-\alpha)}\right\}^{\frac{1}{\alpha}}.$$