On the population density distribution across space: a probabilistic approach

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Abstract

This paper develops a novel approach to study the distribution of the regional population density across space. We work in a Bayesian parametric framework. By exploiting the Gamma distribution, we are able to introduce heterogeneity across space without incurring any a priori definition of territorial units. Our contribution also permits the inclusion of an approximation of individual preferences as a further driving force in location choices. We perform an empirical application to the case of Massachusetts. Our results demonstrate that a subjective measure of distance performs well in replicating the population distribution across Massachusetts.

Keywords: Agglomerations, Bayesian inference, Distance, Gibbs sampling, Kendall's tau index, Population density.
JEL Classification: C11, C51, R10, R14; **AMS Classification:** 62J12, 62P20.

1 Introduction

Explorations of the determinants of the territorial organization have flourished in the current economic literature. An understanding of these factors provides new insights that are extremely useful for decision-makers in planning interventions that better fit agents' expectations. The complexity of this type of analysis lies in the difficulty of fully grasping the way in which these features combine to produce an evident unequal distribution of the population or activities across space. In this respect, the concept of accessibility turns out to be fundamental in proposing strategies to control for an uneven space structure. The accessibility criterion introduces heterogeneity across space: not all locations can be considered as equivalent from a strictly economic viewpoint. According to the accessibility concept, people enjoy easy access to all of the amenities and other facilities that they desire (Fujita and Thisse, 2002; or Song, 1996).

They show a particular interest in locating themselves as close as possible to the place (usually identified as a central business district) that guarantees easy access and proximity to these amenities.

Whenever consumers prefer one location over another, their distribution across space is not expected to be uniform. Several exogenous parametric methods have been proposed to measure the geographical accessibility by adopting a sample of selected distance functions but with evident limitations in the generalization of the results. In this respect, the analysis we are proposing aims at providing a novel framework for the study of the population density distribution in a regional setting. The regional dimension embeds an endogenous territorial heterogeneity by accounting for both rural and urban plots. Centering on the case of monocentric distributions, our idea is to provide an as general as possible setting for drawing a population density distribution in a dynamic framework in which the size of the regional water and urban plots may also vary.

Our approach applies to the case of monocentric density. This method of analysis follows the guidelines proposed by Nairn and O'Neill (1988). As we will illustrate in Section 2, the probabilistic approach we are designing provides an interesting framework to overcome the limits associated with the exogenous choice of a distance function. We do not define any a priori structure of the spatial territory, instead allowing our framework to be suitable for adaptation to the largest range of settings. In the same spirit as Helsley and Strange (2007), we also do not define an ad-hoc deterministic distance function to represent the accessibility concept. We model instead both population density and distance functions as random continuous variables. Our strategy is to conceptualize a representation of a subjective measure of distance embedding subjective preferences (e.g. race, social networks, cultural proximity or similar determinants as discussed in Zenou, 2009) influencing location choices beyond the canonical physical distance. These individual preferences can affect the economic distance between two geographic locations with respect to a common center even if the Euclidean (namely physical) distance between these locations is identical. By considering distance as a random variable, we can model a broad spectrum of other discriminating factors that make distance a subjective concept for each single individual rather than an objective concept and therefore, generate spatial heterogeneity. Then, that measure will be the key tool in the understanding of the distribution of the population density across space. We focus on the population density distribution rather than on the usual conditional expectation to better control for heterogeneity by capturing the true attractiveness of points in space at and being able to provide point estimations.

In terms of application, our method has been implemented to estimate the regional population density in Massachusetts by exploiting both town and census tract data. To be able to replicate the population density distribution in Massachusetts, we construct a setting for which the novelty is to be founded on the association of individual preferences (concerning location decisions) as the main determinant driving population distribution. In this setting, preferences are modeled according to a combination of several factors picturing the attractiveness of a location. The multiple combinations of those factors are the source of an important non-constant heterogeneity that we must control for in the estimation procedure. Furthermore, our selection also entails the embedding of a degree of spatial dependence into the definition of 'joint' preferences for the location of agents living in the territorial unit that we need to control for.

From a technical viewpoint, we are able to estimate the determinants of the population density distribution using a Bayesian Gamma model, via a type of Gibbs sampler method, and, the spatial heterogeneity issue is addressed by accounting for some Gamma county random effects (quite similar to the empirical strategy followed by Low and Hining, 2004).

In the econometric exercise that ends the paper, the proposed technique succeeds in fitting not only the mean of the true population density but also its behaviour in the upper-tail; indicators of subjective preferences (e.g. income, education, age and ethnical composition) are revealed as key determinants allowing for the fit of the model to be improved with respect to the baseline specification including exclusively physical distance as a covariate. In addition, the joint magnitude of the subjective-preference variable coefficients is not negligible with respect to that of the physical distance.

The remainder of this paper is organized as follows. Section 2 presents a literature review and describes our analysis strategy. Section 3 develops our analysis setting and illustrates a Gamma-Gamma econometric model with some of its relevant properties. Section 4 applies the model to the Massachusetts case, and Section 5 concludes. All proofs and other materials are included in the Appendix.[1]

2 Literature review and analysis strategy

The study of the sources of heterogeneity in the population density distribution is conducted at two levels: urban and regional. The former environment focuses on the study of the distribution of the population in a city in relation to the changes incurred in its urban structure. The study of the urban population density is a quite old concern in urban economics: population density distribution patterns help in understanding economic and social matters at urban level. The seminal paper by Clark (1951) is the first attempt to provide a rigorous and formal setting for tackling the previous issues. Since then, interesting empirical methods (grounded to theoretical frameworks) have been developed. In Muth (1969) and Mills (1972) there appears a clear concern for estimating and tracking over time and across space the changes in population density (at urban level) and the decline of density per unit of distance. Other important contributions addressed the problem to get unbiased estimations when working with population density. We recall McDonald and Bowman (1976), Frankena (1978) or Alperovich (1982) among the others. According to Anas et al. (1998), the first and simplest approach is to work with a monocentric density function. In this case, land use is very simple and the negative exponential function helps to explain the strongest empirical regularities. The setting becomes more complex when we refer to polycentric cities that include a central metropolitan area and large subcenters (McMillen, 2004) and some of the attractive characteristics of the metropolitan area still survive in the subcenters.

The regional environment instead addresses the same type of questions in a territory whose distinguishing features are not so uniform as in the previous situation. According to Parr (1985a, 1985b), a region is usually defined as a territory that includes both urban areas and non-metropolitan areas (usually surrounding the metropolitan areas). The importance of studying a regional population density function is that the distribution of people is not identical in the metropolitan and non-metropolitan areas. In particular Parr states that in non-metropolitan areas the density function decreases against the distance at a different rate from the metropolitan areas. Regardless, both approaches (urban and regional) accept the accessibility concept as the key tool to model the population distribution density.

In economic literature modeling of the population distribution across space is associated with an investigation of the accessibility opportunities; the density of population across space is based on the re-

[1] An extended version of this study can be found at http["TBA"] with some complementary material.

lationship between the value of the population density at a given location and the corresponding distance of such a location from a central point, e.g., the central business district (see Nairn and O'Neill, 1988).[2] As listed in Parr et al. (1988) or Song (1996), several possible accessibility functions can be applied, stemming from the variety of distance functions that can be adopted. Commonly, the population density distribution is represented as a discrete function. Then, the population density is the product of an index of population density at the center and an accessibility function that decreases with increasing the distance from that center. For example, the most common applied accessibility function is the inverse of the physical distance from the urban center. Unfortunately, this procedure is very controversial. First the general distance from a center can be measured in multiple ways such that we need to identify the most suitable representation of the distance function for the setting we are considering. Second, population location choices can be driven by racial or ethnic preferences, housing issues, demands for local public goods or other factors that cannot always be properly embedded into a simple distance function (refer to Zenou, 2009 or Glaeser, 2008 for a wider discussion about this issue).

The investigation on the population distribution in the regional case needs to extend the urban setting to include a further dimension: the heterogeneity of space. A regional space is commonly composed of urban and rural land and therefore, we cannot exclusively rely on the proper modeling technique employed for urban areas. Once more, the problem resides in the definition of the distance function. Parr (1985b) has performed an extensive research and defines three characteristics that models aiming at replicating the regional population density criteria must fulfill: *i)* the density has to reach a maximum in a short distance from the metropolitan center (which here represents the equivalent of the urban central business district), *ii)* the density should approximate the negative exponential equation between the center and the metropolitan boundary, and *iii)* the function should approximate the Pareto function over the portion from the metropolitan boundary to the regional one. Among the quadratic exponential, linear gamma and lognormal functions that are usually considered as potential candidates, only the lognormal function seems to satisfy these three requirements. Nairn and O'Neill (1988) go beyond this result: by adopting the techniques of differential equations and asymptotic analysis, they are able to achieve a functional form for the typical behavior of a regional population density fulfilling the three previous criteria. Nevertheless, in their conclusions the authors sketch a possible way of generalizing such a function by conceptualizing a probabilistic density function for which skewness -linked to the standard deviation- is a measure of the dispersion of the population with respect to the distance from the regional center. Our starting point develops the last argument put foward by Nairn and O'Neill (1988). As a novelty, we consider both the population density and the distance function as random continuous variables (rather than deterministic variables). The idea of a random distance function updates the basic notion of (deterministic) physical distance by including additional factors that are expected to impact the distance perception of individual agents. In our model, the conditional distribution of the density population (at a given distance) assigns more probability to the larger values of the density population conditioning on lower values of the distance. Our key strategy is to apply the notion of *Negative Regression Dependence* to associate the population density with the distance and we measure the Negative Regression Dependence by means of the Kendall's tau index. This index is an extremely flexible device: it does not change under every

[2] In the urban tradition, Mills and Ping Tan (1980) and McDonald (1989) present interesting surveys of the most common population density function studies in the literature whereas Bracken and Martin (1988) review the spatial population distribution for census data.

monotone deterministic transformation of the density population or the distance. Moreover, it can be calculated even if the moments of the density population do not exist. Since we expect the density population variance increases in the mean and the density is highly skewed, then we opt for a Gamma model. In particular, we assume that the population density is gamma distributed conditionally on distance and negatively regression dependent on the distance and that the distance is Gamma distributed (Gamma-Gamma Model). In this particular case, we can effectively evaluate to what extent and how rapidly the density falls off with distance in terms of the values of the parameters of the gamma distributions.

Our study completes with an empirical application of our method to the population density in Massachusetts. This case study represents a suitable situation for replicating the monocentric framework, taking Boston as a neuralgic regional center and characterizing the individual preferences of agents with the common goal of settling close to this center. Actually, as also discussed in Glaeser (2008), since Boston is the financial and cultural heart of Massachusetts, it seems quite likely and reasonable that people will display clear preferences for settling in its proximity.[3]

Another novelty of this contribution is the exploitation of the Bayesian parametric framework to perform our econometric analysis. The Bayesian approach is not very commonly used in analyzing this type of topic. As discussed in Anselin (2010), LeSage and Pace (2009) (and previously LeSage, 1997) propose an interesting contribution in terms of the potential applications of the Bayesian approach in spatial econometrics. The Bayesian approach is quite widespread in statistics: it allows for departing from the standard model assumption and controlling for dependence and heterogeneity in the same framework. In this respect, as a novelty in the case of spatial analysis, we adopt the frailty method as a tool to speed up the convergence process. In particular, the Bayesian method provides robust estimations in cases of important outliers (which are lessened in their influence) and the introduction of subjective prior information allows for the control of the potential problems of "weak data". In this respect, the non-parametric methods suffer from problems with non-constant variance over space and the "weak data" puzzle arises from the adoption of a distance-weighted sub-sample of neighboring data to produce linear regression estimates for each point in space. Quantile regression might be another way to deal with this problem, but additional investigation on the most preferable approach is left for further investigation.

In our sample we need to control for the distortion of heterogeneity and the (potential) spatial dependence. A simple description of the population distribution across Massachusetts exploiting town data confirm our concerns: we need to treat a clear uneven territorial distribution (See Figures 4 in Appendix A) with several important density picks.[4] An extremely important population-concentration selection appears in the correspondence of the Boston area and others of different magnitude are irregularly scattered across the territory. In the light of this evidence, in our opinion, focusing on the population-density distribution (namely focusing on the density index) is a convenient choice because this indicator allows for a better control of the spatial variation effect.

[3]In this line we remember the contribution of Rappaport (2009). He showed that the US population has been migrating to places in search of amenities accompanied by increasing wealth.

[4]The figure of the population density distribution across census tracts is available upon request

3 Modeling preferences and probabilistic distribution

Our framework stems from the idea that the spatial distribution of population across a regional territory is the result of the combination of the subjective preferences of residents of that region with respect to the place they want to settle down. The agents' choice is often based on economic reasons or historical and cultural heritage on which all individuals belonging to the same territorial unit agree (Parr, 2007). In a previous stage (not considered here) each citizen defines his or her location choice by maximizing his or her own utility function under a budget and other possible constraints. Hence, once this step is accomplished we need to define a function representing the distribution of these individuals taking into account their location preferences. To do so, we identify one-dimensional space (region) with the continuous line $X \in (0, \infty)$ and the total surface of land in each location $x \in X$ equal to one. We consider a continuum N of heterogeneous agents (or households) who choose to reside in that space (Fujita and Thisse, 2002). In this specific framework, we work under the assumption that the agents differ in their degree of preference to set close or far from a business center (CBD).[5] In particular, the CBD is defined as a place in which agents either do or do not have an interest for. For the sake of simplicity, we focus exclusively on the modeling of the case in which all the individuals aspire to settle as close as possible to an exogenous selected CBD,[6] and assume the following individuals agents' preferences.

Assumption 1 *Given two locations $(x, z) \in X$ with $0 < x < z$, then each agent prefers x to z.*

In our setting, the one-dimensional space X is the set of location alternatives allowing for the comparison of two pairs of alternatives, namely two different location points. According to Mas-Colell et al. (1995), to satisfy a certain degree of consistency in individual agents' choices (here, the individuals' preferences versus their residence lots), we have to assume that their choices satisfy the weak axiom of revealed preferences. Without loss of generality, we define the CBD at zero and X as the distance function from the CBD. In modeling the idea of subjective preferences, the agents' perception of spatial distance has to be interpreted as a combination of many different factors: environmental conditions, cultural heritage etc, beyond the physical distance from the CBD. This way of modeling the distance function allows us to introduce the discriminatory degree of preferences of all the individuals.

To complete the setting, we need to define the way we aggregate the individual choices to shape the population distribution. Let Y denote the population density in the space and $F_{Y|X}$ be its cumulative distribution function conditional to the distance X: $F_{Y|X}(y \mid x) = P(Y \leq y \mid X = x)$. Assume that

Assumption 2 *Y is* negatively regression dependent on *X, i.e.*

$$F_{Y|X}(y \mid x_1) \leq F_{Y|X}(y \mid x_2),\ \forall y \in \mathbb{R} \text{ and } \forall x_1 < x_2\ . \tag{1}$$

Inequality (1) says that the proportion of territorial units at a distance x_1 from the CBD with a population density at most equal to y is no greater than the proportion of the more distant territorial units (x_2) with

[5] We assume that the consumers preferences also encompass the easy assess to economic activities as well as employment centers. Because we do not control them explicitly, they will contribute to the gamma random effects in the model described in Section 4.1.

[6] This framework could be symmetrically adapted to the case in which an individual seeks to settle far from a specific exogenous place. Instead, problems may arise when the variance across their preferences is extremely large. For a discussion about this issue refer to Fujita and Thisse (2002).

a population density at most equal to y. According to Assumption 2, large distances X from the CBD tend to be associated with small densities of population Y and the hypothesis we introduce regarding agents' preferences implies that the population density distribution is likely to decrease as the distance from the CBD increases. Thus, Assumption 2 is the translation in a probabilistic setting of Assumption 1, representing both the idea that all citizens prefer to settle close to the CBD.

Negative regression dependence is an asymmetric concept: the fact that Y is negatively regression dependent on X does not imply that X is negatively regression dependent on Y. Furthermore if Y is negatively regression dependent on X, then $g(Y)$ is negatively regression dependent on $h(X)$, for every increasing functions g, h. In other word, negative regression dependence is robust with respect to every increasing deterministic transformations of X or Y or X and Y. Finally, the fact that Y is or not negatively regression dependent on X is a nonparametric question, in the sense that the concept of negative regression dependence applies whatever is the parametric assumptions on the distribution of X, Y. [7]

A non-parametric way of measuring the degree of negative regression association of Y on X is given by the Kendall's tau index.

Definition 3 *The* Kendall's tau index τ *of a random couple* (X, Y) *is*

$$\tau = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$$

where (X_1, Y_1), (X_2, Y_2) *are two independent copies of* (X, Y). *If the random couple* (X, Y) *is continuous, then* $\tau = 1 - 2\pi_d$, *where* $\pi_d = P((X_1 - X_2)(Y_1 - Y_2) < 0)$.

The Kendall's τ index assumes values in interval $[-1, 1]$ and is negative if and only if $\pi_d > 1/2$. If τ is negative, random variables X, Y are called *discordant or negative associated*: τ is negative (if and) only if the probability that either $x_1 < x_2$ is associated with $y_2 > y_1$ or $x_1 > x_2$ with $y_2 < y_1$ is greater than 1/2, meaning that X and Y are discordant if their values are dissociated with a high probability, namely greater than one half. On the other hand, we prove in Appendix B that *if Assumption 2 is true then* X, Y *are discordant, i.e.* $\tau < 0$.

Remark 4 Similar to the negative regression dependence, the value of the Kendall's τ of a couple (X, Y) also does not change under every increasing monotone deterministic transformation of X or Y or X and Y. In fact, $P(X_2 > X_1, Y_2 < Y_1) = P(g(X_2) > g(X_1), h(Y_2) < h(Y_1))$, for any increasing function g, h. This remark turns out to be very useful first to simplify the computation of the Kendall's τ and second to shed light on what parameters effectively determine the Kendall's τ and thus influence the dependence of the density population Y from the distance to CBD X, once a joint model for X, Y is assumed.

[7] From a historical point of view, the concept of negative regression dependence was introduced by Lehmann (1966). More recently, the "positive" regression dependence has been used in insurance theory to model the dependence between the insurable and uninsurable risk (Dana and Scarsini 2005). Moreover, the positive regression dependence is a notion similar to that of "affiliation" discussed in Milgrom and Weber (1982) and used in auction theory. As demonstrated in Dana and Scarsini 2005, affiliation implies positively regression dependence.

From an economic viewpoint the concept of *negatively regression dependence* identifies with an operative device to fully shape the preferences of individuals when referring to their location choices inside in a regional territory.

By referring to the Kendall's τ index, we are able to provide a measure of agents' preferences in terms of the location choices.[8] Therefore, the use of the Kendall's τ allows us to discriminate between the distribution functions that can be suitable for our framework of analysis and all of the other distribution functions.

How much X, Y are discordant depends on joint distribution for the distance and the population density. There is no one way to define a priori population density and spatial distance distributions: any positive random variables for which Assumption 2 holds can be selected. To provide the flavor of this selection technique, we focus on a couple of functions suitable to represent the population density distribution across a regional space: the following log-normal and gamma models were already discussed in Parr at al. (1988) and Song (1996).

Example 5 (Log-normal Model) Let Y define as

$$\ln Y = \alpha_0 - \alpha X + \epsilon \tag{2}$$

where ϵ is a random disturbance term with zero mean and constant variance and X and ϵ are independent. Then Y is negatively regression dependent on X as $\alpha > 0$. In fact, the conditional cumulative distribution function of Y given $X = x$ corresponds to that of $\exp\{\alpha_0 - \alpha x + \epsilon\}$ which clearly stochastically decreases in x if $\alpha > 0$. In Equation (2) the density of population Y is modeled as a negative exponential function of the distance from a territorial center (e.g., a CBD) and the parameter α is the density gradient that describes how rapidly the density falls off with distance. This corresponds to a classical analysis of the accessibility problem where, by assumption, α is assumed to be greater than zero. Refer, for example, to the estimation function of the accessibility measure numbered one in the first row of Table 1 in Song (1996). Thus, this example emphasizes that the classical log-normal regression analysis of the location accessibility satisfies Assumption 2.

Example 6 (Gamma-Gamma Model) Let $\eta(x) : (0, \infty) \mapsto (0, \infty)$ be a monotone increasing function and suppose that conditional on $X = x$, Y has Gamma distribution[9] with shape θ and rate function $\theta \cdot \eta(x)$:

$$Y|X = x \sim \Gamma\left(\theta, \theta\eta(x)\right) \ . \tag{3}$$

Thus Assumption 2 is satisfied. For technical details see Appendix C. The conditional mean of Y given $X = x$ is $\mathrm{E}(Y|X) = 1/\eta(X)$ and the conditional coefficient of variation CV is constant and

$$CV = \sqrt{\mathrm{Var}(Y|X)}/\,\mathrm{E}(Y|X) = \sqrt{\mathrm{E}^2(Y|X)/\theta}/\,\mathrm{E}(Y|X) = \frac{1}{\sqrt{\theta}} \ .$$

If $X \sim \Gamma(\alpha, \beta)$ with $\alpha, \beta > 0$, we shall denote the joint model: $Y|X \sim \Gamma\left(\theta, \theta\eta(X)\right)$ and $X \sim \Gamma(\alpha, \beta)$ as

[8] In the spirit of Parr (1985a) we are interested in working within a framework that is suitable to shape the population density distribution for territories including both urban and rural areas. This reason explains the decision to focus on the regional rather than the urban dimension.

[9] A Gamma probability density with shape α and rate β has positive support and its probability density on $(0, \infty)$ is

$$\frac{\beta^\alpha x^{\alpha-1} \mathrm{e}^{-\beta x}}{\Gamma(\alpha)},$$

where $\Gamma(\alpha)$ represents the Gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-\beta x}\, dx$.

$\Gamma(\theta,\theta\eta(X))\times\Gamma(\alpha,\beta)$. Two interesting models correspond to the following alternative choices for the link function $\eta(x)$: $\eta(x)=ax+b,\ a>0$ and

$$\eta(x)=\frac{\mathrm{e}^{aX}}{b}\quad a,b>0, \tag{4}$$

In the light of Remark 4 and of the properties of the Gamma distributions family, we have that the models $\Gamma(\theta,\theta aX)\times\Gamma(\alpha,\beta)$ and $\Gamma(\theta,\theta X)\times\Gamma(\alpha,1)$ share the same Kendall's τ.[10] It follows that, the Kendall's τ of a Gamma-Gamma model $\Gamma(\theta,\theta aX)\times\Gamma(\alpha,\beta)$ depends only on the shape parameters α and θ and, is independent of both the two rate parameters a and β.

For $\eta(x)=ax+b$, Gamma-Gamma models $\Gamma(\theta,\theta(aX+b))\times\Gamma(\alpha,\beta)$ and $\Gamma\left(\theta,\theta(a/\beta X+b)\right)\times\Gamma(\alpha,1)$ share the same τ. In other words, if $b\neq 0$ then τ depends on the shape parameters θ, α and the gradient a/β computed as a pure number: without loss of generality, in our numerical calculations of τ for a Gamma-Gamma model with $\eta(x)=ax+b$, we can let $\beta=1$.

If $\eta(x)$ is an exponential function as in (4) then

$$\tau(X,Y)=\tau\left(\beta X,\frac{Y}{b}\right)=\tau(W,Z),\qquad \text{where } W\sim\Gamma(\alpha,1),\ Z|W\sim\Gamma(\theta,\theta\mathrm{e}^{aW/\beta})\ .$$

Additionally, in this case, τ depends only on the coefficient of variation $\mathrm{CV}(Y|X)=\theta^{-1/2}$, on the shape α of X and on the ratio a/β. As a consequence, in our numerical calculations of τ for a Gamma-Gamma model with link function (4), we can set $\beta=1$.

As a matter of fact, the expression for the Kendall's τ of a Gamma-Gamma model can only be evaluated numerically or via simulation. Some simplifications for τ arises when α and θ are integers. For example, when $\eta(x)=ax$ and $\alpha=\theta=1$, one has

$$\tau(X,Y)=2\int_0^\infty\int_{x_1}^\infty\int_0^\infty\int_{y_2}^\infty x_1x_2\mathrm{e}^{-x_1y_1-x_1-x_2y_2-x_2}\ dy_1dy_2dx_2dx_1-1=-0.5\ .$$

In general, to evaluate τ we simulated $N=20\ 000$ independent random couples $\{(X_i,Y_i)\}_{i=1}^N$ from the distribution of (X,Y) and calculated the *empirical Kendall's coefficient of concordance*:

$$R_K=\frac{C-D}{N(N-1)}$$

where C is the number of "concordant" couples and D the number of "discordant".[11]

Table 1(a) shows the results of the simulations for the linear link function $\eta(x)=ax+b$ and Table 1(b) for the exponential $\eta(x)$ in (4). One observes in Table 1(a) the following: *i*) for fixed b and θ, the negative

[10] In fact, if $W=\beta X$ and $Z=mY$ with $l,m>0$, then $f_W(w)\sim\Gamma(\alpha,1)$ and

$$f_{Z|W}(z|w)=\frac{1}{m}\times f_{Y|X}\left(\frac{z}{m}|\frac{w}{\beta}\right)=\frac{\left(\theta\eta\left(\frac{w}{\beta}\right)/m\right)^\theta}{\Gamma(\theta)}z^{\theta-1}\exp\left\{-z\frac{\theta}{m}\eta\left(\frac{w}{\beta}\right)\right\}\ .$$

so that the Kendall's τ of a $\Gamma(\theta,\theta\eta(X))\times\Gamma(\alpha,\beta)$ model is equal to the Kendall's τ of a $\Gamma(\theta,\theta\eta(X/\beta)/m)\times\Gamma(\alpha,1)$ model and, for $m=a/\beta$ we provide the result.

[11] Two pairs are *concordant* if both members of a couple are larger than their respective members of the other couple, whereas two pairs are *discordant* if the two members of one couple differ in the opposite sense from the respective members of the other couple.

(a) The case $E(Y|X) = 1/(aX + b)$.

a	b	$CV(= 1/\sqrt{\theta})$	α = 1	10	50	100
		0.2	-0.889	-0.646	-0.392	-0.289
0.5	0	1.0	-0.499	-0.175	-0.079	-0.063
		4.0	-0.081	-0.023	-0.011	-0.009
		0.2	-0.582	-0.580	-0.385	-0.292
0.5	1.0	1.0	-0.147	-0.148	-0.078	-0.067
		4.0	-0.021	-0.019	-0.006	-0.013
		0.2	-0.127	-0.308	-0.299	-0.247
0.5	10.0	1.0	-0.018	-0.057	-0.063	-0.058
		4.0	0.000	-0.009	-0.006	-0.006
		0.2	-0.888	-0.648	-0.381	-0.297
1.0	0	1.0	-0.504	-0.175	-0.071	-0.058
		4.0	-0.078	-0.019	-0.013	0.002
		0.2	-0.704	-0.614	-0.382	-0.294
1.0	1.0	1.0	-0.227	-0.160	-0.074	-0.058
		4.0	-0.023	-0.011	-0.011	-0.010
		0.2	-0.237	-0.413	-0.337	-0.264
1.0	10.0	1.0	-0.047	-0.098	-0.065	-0.052
		4.0	-0.008	-0.005	-0.006	-0.005
		0.2	-0.888	-0.647	-0.396	-0.289
10.0	0	1.0	-0.507	-0.174	-0.072	-0.052
		4.0	-0.077	-0.023	-0.004	-0.005
		0.2	-0.866	-0.643	-0.387	-0.295
10.0	1.0	1.0	-0.433	-0.175	-0.072	-0.058
		4.0	-0.049	-0.015	-0.006	-0.005
		0.2	-0.709	-0.614	-0.387	-0.297
10.0	10.0	1.0	-0.233	-0.157	-0.072	–0.053
		4.0	-0.029	-0.021	-0.011	-0.010

(b) The case $E(Y|X) = e^{-aX}$.

a	$CV(= 1/\sqrt{\theta})$	α = 1	10	50	100
	0.2	-0.670	-0.916	-0.963	-0.974
0.5	1.0	-0.233	-0.579	-0.788	-0.846
	4.0	-0.026	-0.105	-0.205	-0.266
	0.2	-0.810	-0.958	-0.981	-0.987
1.0	1.0	-0.388	-0.759	-0.890	-0.923
	4.0	-0.056	-0.177	-0.356	-0.436
	0.2	-0.978	-0.996	-0.998	-0.028
10.0	1.0	-0.878	-0.975	-0.988	-0.041
	4.0	-0.387	-0.737	-0.873	-0.030

Table 1: Kendall's τ for the Gamma-Gamma model in Example 6; Table (a) refers to the case $E(Y|X) = 1/(aX + b)$ with coefficient of variation $CV = 1/\sqrt{\theta}$ and $\beta = 1$. (The variability in Kendall's τ when $b = 0$ for different values of a –for example, $\tau = -0.499, -0.504, -0.507$, for $\theta = 1$ and $a = 0.5, 1, 10$, respectively– is exclusively due to the simulation errors). Table (b) refers to the case $E(Y|X) = e^{-aX}$ with $\beta = 1$.

dependence decreases as α increases and the Kendall's τ approaches zero from below as $\alpha \to \infty$; *ii*) given the shape parameter α of the Gamma distance X, the negative dependence decreases with the conditional coefficient of variation $\text{CV} = 1/\sqrt{\theta}$ of Y given X. Actually, larger values of CV imply larger dispersion for Y, whereas larger values of α imply a larger variance for X and, thus a smaller dependence between X and Y in both the cases. Finally, given θ, $b(\neq 0)$ and α, the dependence between X and Y increases as the gradient a increases and is more relevant with higher values of b and θ are. As shown in Table 1(b), the same findings hold in the case of a Gamma-Gamma Model with the exponential link function $\eta(x)$ in (4).

Measuring the degree of dissociation in the location choices of agents by means of τ, we note that the negative dependence proportionally reinforces the shape θ, i.e. when the distribution density polarizes. In this sense, the parameters of the function replicate the degree of skew of the density and allow the replication of the different gradient of the density across space.

Taking into consideration these land dynamics, whenever the territorial center of reference enlarges, the preferences of agents coincide because they all want to settle close to this center, and this situation occurs for any distance function shaping the space. From a regional perspective, the enlargement of the CBD results in more blurred agents' preferences with respect to a possible location close to the CBD, above all when the shape parameter of the distance function α increases. This is due to the polarization of the distance function that makes this spatial dimension disappear. Referring to some empirical evidence (e.g., Chicago as in McMillen 2004), the enlargement of the central business district corresponds to the creation of subcenters (which maybe driven by some external factors such as firm location) surrounding the central metropolitan area. The new subcenters reinforce the size and attractiveness of the urban area (namely the CBD in our framework) with respect to the rural area. Therefore, from a regional perspective, an extreme case would be to obtain an ever enlarging center up to the point, where the regional space is reduced to a single urban point without a water plot.

A model like the Gamma-Gamma one –with constant coefficient of variation– is very useful if we expect the variance of Y to increase with its mean.accessibility as does the distance from a center. Once more, the parameter a is a measure of the density gradient describing the decreasing speed of the density with respect to the distance, whereas the parameter $\ln b$ describes the density at or near the CBD. In other words, the negative dependence of the density population from the distance from the CBD does not depend on b, as we observe before. Nevertheless, if we take a log link, i.e. $\ln \text{E}(Y|X) = \ln b - aX$, then Model (3)-(4) can be analyzed under the generalized linear model setup (see McCullagh and Nelder, 1989).

Both the Log-normal model at (2) and the Gamma model in (3)-(4) have a constant coefficient of variation and thus both are often useful in similar problems. Using the Gamma Model (3)-(4) to address the accessibility is equivalent to adopting a regression model with multiplicative gamma errors for the original data, whereas Log-normal Model (2) provides an additive regression model for the logarithms of density. In other words, in a Gamma model, the density population is measured on the original scale, whereas in a Log-normal model the scale of measurement of the density is logarithmic. McCullagh and Nelder (1989) suggest the assumption of a Gamma distribution if it is preferable to work with data in original scale (an issue also discussed in Parr, 1985a). Such is the case if, with data on the spectrum $Y_1, \ldots, Y_n$, the sum $Y_1 + \cdots + Y_n$ has a physical meaning. Actually, in our case study in Section 4, the sum of population densities $Y_1, \ldots, Y_n$ of all the n territorial units of space (alternately: census tracts and towns in Massachusetts) is the density population of that space. Furthermore, Firth (1988) proved that, under reciprocal misspecification, a gamma model performs slightly better: analyzing lognormal data with a gamma model is more "efficient" than analyzing gamma data assuming log-normality.

4 A case study for the Gamma-Gamma Model: the population density in Massachusetts

To illustrate the properties of our framework, we propose an empirical application to the case of Massachusetts. Data are taken from the US Census Bureau and refer to the year 2000. We alternately consider census tract and town data to assess the robustness of our novel approach above all when increasing the sample size to encompass a larger degree of spatial variation. To quantify the available information, the database dealing with census tract territorial unites includes 1361 entries, whereas the town data focus on 351 towns belonging to the state. They both are grouped by the 14 counties.

In our monocentric theoretical framework, first of all we tackle the issue of identifying the CBD. To this end, we proceed by selecting the 14 county-seats in Massachusetts as the most representative centers and suitable to be considered as preferred locations and we compute the bilateral distances from *i)* any central points of the 1361 census tracts and *ii)* any town between the 351 in our sample.[12] We then consider (in alternation) each of 14 (Euclidean) distances as the unique explanatory variable and run a reduced form of our regression Model (9) that we label *Model 0*. We perform the estimations using the set of census tract and town data. Of the 14 models, we select the "best match" of the distribution of the population density, according to the Bayesian Deviance Information Criterion (DIC): the model yielding the smallest value of DIC is chosen.[13] We summarize the DIC results obtained for CBD selection in Table 2, from which we deduce that the distance from Boston ensures the best fit to the real data.[14] In the wake of the current literature, this result is not totally unexpected. Empirical evidence does suggest that the geographic center of reference in Massachusetts is the capital Boston. Parr (2007) asserts that Boston can be considered as an example of a "Built city" because of its relative importance for Massachusetts as a focus of economic activity and a demand for labor. Therefore, we choose Boston as our center to carry out our empirical analysis.

In Table 5 in Appendix A we elaborate some descriptive statistics at the town level. First of all, we recognize in Table 5 a negative association between the population density and distance from Boston: a lower population density is associated with a greater distance from Boston. Here the distance is understood as the shortest distance from each county to Boston. Nevertheless, the simple physical distance from a selected (attractive) CBD is not the unique factor shaping the population distribution across space. Other factors contribute in a sensitive way to define the preference of citizens to settle in specific

[12] These distance values have been computed by using the ArcView GIS program. In performing the selection of the census tract of the 14 county-seats we are following the methodology proposed by McDonald (1987).

[13] The DIC is intended as a generalization of Akaike's Information Criterion (AIC); DIC is given by the deviance, (i.e., minus two times the likelihood) calculated at mean of the posterior distribution of all parameters plus two times the "effective numbers of parameters": pD. Models with a lower DIC have to be preferred over those with a larger DIC.

[14] If we accounted for models different in DIC less than 5 with respect to the lowest DIC, other potential CBD would have to be taken into consideration. It is the case of Quincy (for town data) and, Lowell for census tract. We performed estimations with Quincy and Lowell as CBD and results are very similar to the Boston ones (in particular with respect to the significance of the covariates), but with a slight lower fit. These results are available upon request.

	Town		Census tract	
	pD	*DIC*	pD	*DIC*
Boston	21.2	**461.5***	19.3	**7043.6***
Greenfield	17.5	536.5	34.0	7162.8
Brockton	19.9	486.9	19.6	7130.8
Lowell	16.8	523.9	19.5	7044.7
New Bedford	19.9	522.2	23.6	7148.9
Pittsfield	23.0	532.8	31.0	7166.2
Springfield	16.6	538.3	27.3	7165.1
Lym	22.1	467.3	18.5	7162.3
Amhrest	17.6	537.7	35.1	7163.1
Edgartown	24.9	529.7	29.1	7155.5
Worcester	15.8	534.4	21.5	7170.4
Quincy	21.0	462.7	19.2	7050.9
Barnstable	31.8	523.7	29.3	7147.7
Nantucket	27.2	528.9	36.6	7162.0

Table 2: Center selection

locations. In this respect, Wood and Parr (2005) have already introduced the concept of *transaction space* in which they discuss how space can influence the coordination issues among agents (or firms) in the presence of a spatial differentiation in institutional, cultural or language characteristics. Hence cultural proximity or infrastructure quality can affect the economic distance between two places, even if the Euclidean distance between them is identical. In our theoretical framework we aim to recover this wider concept, and which is the reason we seek to identify a number of predictors of the population density in addition to the physical distance. We argued that population location preferences are likely to include a number of environmental characteristics as a discriminating component of the agents'preferences in terms of location choices. We then augment the traditional monocentric regional density model (as in Beckman, 1971) centered on the decay effect of distance ("$Dist$") by adding the ethnic ("Mix"), age ("Age") and education ("$Educ$") composition as well as the income ("Inc") level of the population living in each territorial unit which we refer to. Our selection relies on the main outcomes of the social network literature in spatial economics (as discussed in Zenou, 2009 or Helsely and Zenou, 2011): individuals often demonstrate a tendency to cluster with other people sharing, for instance, the same culture, or with a similar income level.

Furthermore, the econometric model we introduce includes some unobserved state variables $X_1, \ldots, X_{14}$ –one for each county– to capture the relationship between the census tracts (or towns) of the same county and, the heterogeneity among the counties. In some sense, X_i summarizes all the predictors of the population density, either observable but neglected, or unobservable, common to all the census tracts (towns) in the ith county.

With regard to the probabilistic assumption on the law of the population density Y, looking at the empirical means and the standard deviations of the population density (and income) across counties of Massachusetts in Table 5 in Appendix A, we note an increase in the standard deviation with larger mean values. This feature of the data makes a Gamma model a good candidate for the adoption because it is characterized by a constant coefficient of variation. Next section will describe the proposed econometric two-stage Gamma-Gamma model.

$V^{(2)}$	$V^{(3)}$	$V^{(4)}$	$V^{(5)}$
Mix	Inc	$Educ$	Age

Table 3: Legenda of local covariates in Model (5)

4.1 The Gamma-Gamma Model

Let Y_{ij} be the density of the population of the jth census tract (or, respectively, town) within the ith county, D_{ij} its distance from the selected CBD (here Boston), $V_{ij}^{(s)}$ the local covariates named in Table 3 and Z_i the size of free land (namely the water areas) in county i. We propose the following model:

$$\begin{aligned} Y_{ij}|X_i &\sim \text{Gamma}\left(\theta, \theta X_i e^{-\left(a_i D_{ij} + \sum_{s=2}^{5} \beta_s V_{ij}^{(s)} + \sum_{s=3}^{5} \beta_{3+s} V_{ij}^{(s)} D_{ij}\right)}\right) \quad \text{independent for all } j = 1, \ldots, n_i \\ X_i &\sim \text{Gamma}(\alpha, \alpha B_i) \quad \text{independent for all } i = 1, \ldots, k \\ a_i &= \beta_1 + b_2 Z_i \text{ and } B_i = e^{b_0 + b_1 Z_i} \ ; \end{aligned} \tag{5}$$

where k is the number of counties of the state and n_i the number of census tracts (or towns) in county i.

The explanatory variable Mix is the proportion of white population in the census tract (town, respectively). We use the concentration of whites to measure the population mix because empirical evidence in Table 5 reveals that many Massachusetts counties have very small non-white population. By including Mix we are implicitly taking into account some location preferences that may be related to ethnic or network features. The idea, as described in Gabriel and Rosenthal (2004) or Zenou (2009), is to represent the greater or lesser willingness of people to interact with each other, even with some practice of discrimination. For instance, we can test to what extend individuals tend to cluster with other individuals belonging to the same ethnicity to share cultural habits.

As the income variable Inc we are alternately adopting the per-capita GDP when running the regression by exploiting the data from the town level and the median household income when focusing on the census tract data.[15] Income represents an interesting feature of the social peer effects and helps to capture the potential presence of evident territorial reality. In this respect the local per-capita GDP is the most representative measure of the territorial reality.

Another possible measure of peer effects is the level of education. Again referring to the social network literature (for instance, Calvo-Armengol, Patacchini and Zenou, 2009), individuals often prioritize the possibility to settle in areas where they share the same cultural interests with their the neighbors. One way of successfully representing this dimension is to refer to the degree of education: persons with a higher degree of education may be more prone to settle in areas with other persons with the same level of education to share common interests and, eventually, fuel joint cultural initiatives. In our econometric exercise, to take into consideration the composition of the education level of the population in a territorial unit, we create a synthetic indicator to take into account the educational composition. Our indicator $Educ$ ranks the territorial units according to the education level of the population. We rank all units first according to the relative frequency of persons (older than 25 years in the year 2000) carrying a high degree and second based on the relative frequency of persons carrying an elementary degree. Then, for

[15]Income data refer to 1999 as recorded in the 2000 Census. For census tracts we select the median household (which closely approximates the per-capita GDP) because per-capita GDP series is missing for census tracts.

each territorial unit, we compute *Educ* as the difference between the higher ranking minus the elementary ranking. In this way, this index allows us to take into account to what extent a unit may emerge as relatively highly educated with respect to other units in Massachusetts.

The covariate *Age* represents the ratio of the population between 18 and 64 years in the year 2000 (namely, working age population) over the total population of each territorial unit. In this respect, this indicator captures the relative concentration of potential workers in a territorial unit of reference.

We complete the analysis by also including in our equation the interaction terms between water area Z, income, education, age and distance from Boston. These interaction terms can capture a degree of potential attractiveness[16] of each selected territorial unit embedding a sort of mass effect for a few select features (smoothed by the physical distance from Boston). They are expected to amplify the attractiveness of a destination privileging one of the environmental priorities that residents are searching for despite the physical distance from the CBD.

Concerning the spatial dimension, the Gamma model in (5) attempts to take into account the spatial dependence in each county by means of the state variables X_i. In our exercise, we are assuming that the land organizational structure of each county is independent of the other counties, but census tracts (or towns) belonging to the same county are characterized by very similar features. The idea is to highlight some common and institutional factors shared by census tracts (towns) inside the county, but these factors may differ across counties as, for instance, the land use regulation. It is also true that the townships belonging to the same county may share specific natural endowments that others township of other counties do not share (e.g, Dukes County is an island). In other words, towns belonging to the same county share some common features that can be associated with fixed effects embedded in Z's and random effects represented by unobserved X's. Therefore, the population densities of different counties behave as independent random variables, whereas the spatial dependence between population densities in the county i is modeled via X_i.

We obtain several features of density of population Y_{ij} through the conditional expectation results and some standard properties of the Gamma distribution. For $\alpha > 1$ we have:

$$\mathrm{E}(Y_{ij}) = \frac{\alpha}{\alpha - 1} \mathrm{e}^{b_0 + b_1 Z_i + b_2 Z_i D_{ij} + \beta_1 D_{ij} + \sum_{s=2}^{5} \beta_s V_{ij}^{(s)} + \sum_{s=3}^{5} \beta_{3+s} V_{ij}^{(s)} D_{ij}} \tag{6}$$

whereas for $\alpha > 2$ we have the following:

$$\mathrm{Var}(Y_{ij}) = \frac{\alpha - 1 + \theta}{(\alpha - 2)\theta} \times \mathrm{E}^2(Y_{ij}) \tag{7}$$

$$\rho(Y_{ij} Y_{ih}) = \frac{\theta}{\theta + \alpha - 1} \ , \tag{8}$$

[16] Enhancing one of the specific characteristics of the peer effect determinants.

since

$$
\begin{aligned}
\mathrm{E}\left(X_i^{-r}\right) &= \frac{\alpha^r B_i^r}{(\alpha-1)\times(\alpha-r)} \qquad \text{for } \alpha > r \text{ and } r=1,2\ , \\
\mathrm{E}\left(Y_{ij}^r\right) &= E(E(Y_{ij}^r|X_i)) \\
&\qquad = \frac{\theta(\theta+1)\cdots(\theta+r-1)e^{r(\beta_1 D_{ij}+\sum_{s=2}^5 \beta_s V_{ij}^{(s)}+\sum_{s=3}^5 \beta_{3+s} V_{ij}^{(s)} D_{ij})}}{\theta^r}\,\mathrm{E}(X_i^{-r}) \qquad \text{for } r=1,2\ , \\
\mathrm{E}(Y_{ij}Y_{lh}) &= \begin{cases} \mathrm{E}(\mathrm{E}(Y_{ij}Y_{ih}|X_i)) = \mathrm{E}(\mathrm{E}(Y_{ij}|X_i)\,\mathrm{E}(Y_{ih}|X_i)) & \text{if } i=l \\ \mathrm{E}(Y_{ij})\,\mathrm{E}(Y_{lh}) & \text{if } i\neq l\ . \end{cases}
\end{aligned}
$$

We learn from (6) that the unconditional expectation of Y_{ij} describes a log-linear regression model that includes the local predictors D_{ij} and $V_{ij}^{(s)}$, the global predictor Z_i and some interaction terms. The variance of Y_{ij} at (7) is quadratic in the mean and the correlation $\rho(Y_{ij}Y_{ih})$ between the population densities of two cities in the same county is always positive and equal for all the cities and the counties. The shape parameters θ and α measure the intensity of the relationship between Y_{ij}, Y_{ih} within counties and introduce the tool for territorial differentiation. In particular, for large values of α, the state variables X_i's are concentrated around a fixed effect $B_i = b_0 + b_1 Z_i$ and, the correlation of the Y_{ij} within counties approaches to zero (see Equation (8)). Similarly, small values of α represent a strong positive relationship of densities among cities in the same county, but a stronger heterogeneity among the counties. For the α value to be equal, the larger the θ, the less concentrated the Y_{ij}'s (see (7)) and the larger the dependence between Y_{ij} and Y_{ih} (see (8)). When the county variable X_i concentrates around B_i, (i.e., only the free land Z_i has a discriminating impact on population distribution within each county) then other kinds of factors have to be considered as potential discriminatory devices (here represented by the parameters shaping the distribution function). Conversely, when those factors are practically identical across census tracts (respectively towns) within the same county, it is less likely that the population density of one census tract (town) will be differentiated. In this second case, the individual preferences are less stringent.

Alternately, we can measure the dependence between the population densities of two census tracts (towns) in a county using the Kendall's τ. In Appendix D we prove that for $\theta = 1$, $\tau(Y_{ij}, Y_{ih})$ is given by $\tau(Y_{ij}, Y_{ih}) = 2/(2+\alpha)$. For $\alpha \to \infty$, both ρ and τ approach zero as $1/\alpha$; but ρ approaches zero faster than does τ. Nevertheless, in Model (5) the dependence between Y_{ij}, Y_{ih} is not linear and thus it could not be detected only by ρ. Finally, the Kendall's τ allows for the detection of the dependence even if α is less than or equal to two, whereas some of the mean values of $Y_{ij}, Y_{i,h}$ involved in Equations (6)-(8) do not exist for $\alpha \leq 2$.

4.2 Likelihood specifications of the Gamma-Gamma model

The next step is to define a suitable representation of the likelihood to estimate the population density in accordance with the statistical framework we have developed. Let now $(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z})$ be the collection of the data $(Y_{ij}, D_{ij}, V_{ij}^{(s)}, Z_i)$ observed for every $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$ and let $\boldsymbol{\beta} = (b_0, b_1, b_2, \beta_1, \ldots, \beta_8)$ be the vector of the 11 regression parameters. The likelihood function $L(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z}; \boldsymbol{\beta}, \theta, \alpha)$ of the parameters $(\boldsymbol{\beta}, \theta, \alpha)$ based on observed data $(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z})$ can be obtained by integrating the conditional joint probability densities of $Y_{ij}|X_i$ over all the random state variables X_i.

In addition, for $\theta \neq 1$, the final form of the likelihood $L(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{Z}; \boldsymbol{\beta}, \theta, \alpha)$ is too complicated to work with.[17] Therefore, it is preferable to consider the alternate model

$$\begin{aligned} Y_{i,j}|w_i &\sim \text{Gamma}(\theta, \theta w_i/\mu_{ij}) \text{ independent for all } j \\ w_i &\sim \text{Gamma}(\alpha, \alpha) \text{ independent for all } i \\ \mu_{ij} &= \mathrm{e}^{b_0 + b_1 Z_i + b_2 Z_i D_{ij} + \beta_1 D_{ij} + \sum_{s=2}^{5} \beta_s V_{ij}^{(s)} + \sum_{s=3}^{5} \beta_{3+s} V_{ij}^{(s)} D_{ij}} \end{aligned} \tag{9}$$

that give rise to the same likelihood $L(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z}; \boldsymbol{\beta}, \theta, \alpha)$. Considering that the random factors X_i and w_i are unobservable, the two models are not distinguishable and every estimate of parameters $(\boldsymbol{\beta}, \theta, \alpha)$ obtained using the likelihood function should be the same. Owing to its increased simplicity we use Model (9) to develop an estimation procedure of $(\boldsymbol{\beta}, \theta, \alpha)$.[18]

In our econometric exercise we estimate four alternative versions of Model (9). The first labeled *Model 0* is a reduced form of (9) where the unique covariate of the regression is the Euclidian distance D_{ij} (i.e. we let $b_1 = b_2 = \beta_2 = \cdots = \beta_8 = 0$ in (9)). We use *Model 0* for two goals: first to identify the CBD (see discussion in first paragraph of Section 4 and Table 2) and second to benchmark the goodness of fit of the augmented model with the other covariates. The other three models employ different indicators of environmental preferences of individuals as explanatory variables of the population density: our first selected model (henceforth referred as *Model I*) focuses on ethnic, age and education indicators and it is obtained by plugging $\beta_3 = \beta_6 = 0$ into Model (9). The second model (henceforth referred as *Model II*) addresses the ethnic, age and income indicators and it is obtained by letting $\beta_4 = \beta_7 = 0$ in (9). Third model (named *Model III*) merges the salient features of the two previous Models *I* and *II* by focusing on all the environmental indicators previously selected and coincides with Model (9). Thus, our idea is to test the statistical robustness of the results by selecting different specifications that either jointly consider or isolate a few of the various indicators and therefore center on specific determinants.

4.3 Prior specifications and Bayesian estimation

We use a Bayesian approach to estimate $\boldsymbol{\beta}, \alpha, \theta$.

From a Bayesian perspective, the unknown parameters are understood as random variables with a *prior* joint distribution, say π and, the statistical problem consists of updating π by computing a *posterior* joint conditional probability of $\boldsymbol{\beta}, \alpha, \theta$, given the data $\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z}$. Then the posterior joint distribution is summarized in a simple way, typically with posterior means giving rise to a point estimate of the unknown parameters. Moreover, the associated posterior standard errors and a $\gamma 100\%$ credible interval[19] for the unknown parameters are computed. Usually, both the joint and the marginal posterior distributions of all the unknown parameters do not have a closed form. Hence one use some Markov Chains Monte Carlo

[17] A representation for $L(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{Z}; \boldsymbol{\beta}, \theta, \alpha)$ is given by $L(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z}; \boldsymbol{\beta}, \theta, \alpha) = \prod_{i,j} y_{ij}^{\theta-1} \times \frac{\theta^{n\theta}\alpha^{k\alpha}\prod_{i=1}^{k}\Gamma(n_i\theta+\alpha)}{\Gamma(\theta)^n\Gamma(\alpha)^k} \times \prod_{i=1}^{k} B_i^{\alpha}\Big(\alpha B_i + \theta \sum_j y_{i,j} \mathrm{e}^{a_i D_{ij} + \sum_{s=2}^{5}\beta_s V_{ij}^{(s)} + \sum_{s=3}^{5}\beta_{3+s}V_{ij}^{(s)} D_{ij}}\Big)^{-(n\theta+\alpha)}$,

[18] The parameter restriction shape = scale = α for the gamma distribution of the frailty terms w_i (which results in $\mathrm{E}(w_i) = 1$) ensures that Model (9) is identifiable.

[19] In Bayesian statistics, a $\gamma 100\%$ credible interval for a parameter η is given by $q_{\frac{1-\gamma}{2}} < \eta < q_{\frac{1+\gamma}{2}}$, where $q_{\frac{1-\gamma}{2}}, q_{\frac{1+\gamma}{2}}$ are posterior quantiles of η.

algorithms to simulate and summarize them.

Let $\boldsymbol{w} = (w_1, \ldots, w_k)$ be the vector of the unobservable "frailties", $(\boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z})$ the set of the "complete data" and $L(\boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z}; \boldsymbol{\beta}, \theta, \alpha)$ the "complete" likelihood given by

$$L(\boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{Z}; \boldsymbol{\beta}, \theta, \alpha) = \frac{\theta^{n\theta}\alpha^{k\alpha}}{\Gamma(\theta)^n \Gamma(\alpha)^k} \prod_{i,j} \left(\mu_{ij}^{\theta} y_{ij}^{\theta-1}\right) \mathrm{e}^{-\sum_{i=1}^{k} w_i (\theta \sum_j \mu_{ij} y_{ij} + \alpha)} \prod_i w_i^{n_i\theta+\alpha-1} \tag{10}$$

with $n = \sum_{i=1}^{k} n_i$. From now on we will work with this complete likelihood (10) and handle unknown frailties $\boldsymbol{w}$ as unknown parameters to estimate.

With regard to the prior, we seleced "non informative" priors for $\boldsymbol{\beta}, \alpha, \theta$ to represent our vague prior knowledge of these factors. A priori $\boldsymbol{\beta}, \alpha$ and θ are assumed to be independent. In particular, the regression parameters in $\boldsymbol{\beta}$ are a priori independent normal random variables with large variances: $b_r, \beta_r \sim \mathrm{N}(0, 10\,000)$, and both the shape parameter α and the rate θ are (independent) exponential with a rate 0.2: this choice for α, θ implies a large prior variance for α, θ.[20]

4.4 Results for the Massachusetts case study

To deal with tractable values, we standardized the regressors, excluding the constant term, by subtracting the sample mean and dividing by the sample standard deviation. All the inferences were coded in JAGS (Just Another Gibbs Sampler) software package by Plummer (2010) which is designed to work closely with the R package in which all statistical computations and graphics were performed. On the whole, 750 000 iterations for three chains were run for the unknown parameters $\boldsymbol{\beta}, \alpha, \theta$ and frailties $w_1, \ldots, w_{14}$, and the first 250 000 were discarded as burn-in. After the burn-in, one out of every 100 simulated values was kept for posterior analysis, for a total of 5 000 simulations saved per chain. Finally we select one final sample among these three. The convergence diagnostics such as those available in the R package CODA were computed (Gelman, Geweke, Heidelberger and Welch stationarity test, interval halfwidth test) for all parameters, indicating that convergence may have been achieved.

4.4.1 The census tract case

In this subsection we consider the data of all the 1361 census tracts in Massachusetts (from the 2000 Census) and report the results of fitting Models *I, II and III.*

Table 4(a) shows the posterior mean, standard deviation (on the first row) and 95% credible intervals (on the second row) for the effective parameters $\boldsymbol{\beta}, \alpha, \theta$. The third row of Table 4(a) shows the posterior predictive p-values of regression parameters: $p_{b_i} = \min\{P(b_i > 0|\boldsymbol{Data}), P(b_i < 0|\boldsymbol{Data})\}$, $p_{\beta_i} = \min\{P(\beta_i > 0|\boldsymbol{Data}), P(\beta_i < 0|\boldsymbol{Data})\}$. We can draw conclusions about the association between each predictors and Y if the p-value is low (for example less than 5%.) Conversely, p_{β_i} (p_{b_i}) is close to 0.5 when β_i (b_i) is concentrated around zero.

Figure 1(a) displays 95% posterior credible intervals of the gamma county frailties $\boldsymbol{w}$, with the value

[20] The most commonly used non informative prior law for a rate parameter is Gamma(ν, ν) with a small ν. However, our exponential prior choice allows for a better mixing of the MCMC algorithm. Moreover, in a sensitivity analysis, we compared the results of fitting the Gamma-Gamma Models using both of these non informative priors and obtained very similar results for the estimates of the regression parameters.

(a) Census tract data

	Model I	*Model II*	*Model III*
b_0 (Intercept)	1.339 (0.173) (0.991, 1.676) 0	1.143 (0.168) (0.796, 1.457) 0	1.199 (0.166) (0.852, 1.307) 0
b_1 (Z_i)	−0.081 (0.020) (−0.119, −0.041) 0	−0.074 (0.019) (−0.111, −0.035) 0	−0.080 (0.019) (−0.118, −0.067) 0
b_2 ($Z_i \times Dist$)	−0.016 (0.021) (−0.058, 0.025) 0.217	−0.009 (0.020) (−0.049, 0.005) 0.330	−0.021 (0.020) (−0.059, −0.008) 0.154
β_1 ($Dist$)	−0.964 (0.101) (−1.163, −0.774) 0	−1.012 (0.087) (−1.187, −0.847) 0	−1.051 (0.092) (−1.233, −0.875) 0
β_2 (Mix)	−0.307 (0.035) (−0.378, −0.239) 0	−0.305 (0.032) (−0.368, −0.243) 0	−0.240 (0.033) (−0.306, −0.176) 0
β_3 (Inc)	−	−0.615 (0.030) (−0.671, −0.556) 0	−0.437 (0.042) (−0.518, −0.355) 0
β_4 ($Educ$)	−0.544 (0.033) (−0.611, −0.479) 0	−	−0.262 (0.043) (−0.347, −0.234) 0
β_5 (Age)	0.441 (0.023) (0.397, 0.488) 0	0.319 (0.023) (0.276, 0.364) 0	0.370 (0.024) (0.323, 0.417) 0
β_6 ($Inc \times Dist$)	−	−0.314 (0.028) (−0.368, −0.261) 0	−0.258 (0.041) (−0.339, −0.176) 0
β_7 ($Educ \times Dist$)	−0.183 (0.026) (−0.233, −0.132) 0	−	−0.044 (0.037)9 (−0.117, −0.01) 0.119
β_8 ($Age \times Dist$)	−0.013 (0.020) (−0.053, 0.027) 0.249	0.008 (0.018) (−0.026, 0.044) 0.337	0.003 (0.018) (−0.032, 0.040) 0.420
$\alpha_{frailty}$	3.059 (1.362) (1.173, 6.467)	3.986 (1.842) (1.437, 8.569)	3.737 (1.751) (1.385, 8.108)
θ	1.546 (0.055) (1.442, 1.658)	1.624 (0.059) (1.511, 1.741)	1.663 (0.059) (1.548, 1.783)
DIC (pD)	6250.3 (30.5)	6169.3 (29.8)	**6136.1***** (33.8)
% outliers	4.34%% (9.18%)	4.26% (9.26%)	4.63% (9.33%)

(b) Town data

	Model I	*Model II*	*Model III*
b_0 (Intercept)	−0.614 (0.213) (−1.074, −0.238) 0.0018	−0.606 (0.202) (−1.047, −0.235) 0.001	−0.618 (0.212) (−1.081, −0.231) 0.0012
b_1 (Z_i)	−0.042 (0.132) (−0.325, 0.204) 0.3848	−0.052 (0.130) (−0.329, 0.183) 0.3504	−0.046 (0.133) (−0.338, 0.187) 0.3706
b_2 ($Z_i \times Dist$)	0.124 (0.142) (−0.126, 0.429) 0.1886	0.102 (0.139) (−0.140, 0.407) 0.2312	0.125 (0.142) (−0.125, 0.429) 0.1886
β_1 ($Dist$)	−1.420 (0.169) (−1.759, −1.099) 0	−1.398 (0.166) (−1.726, −1.087) 0	−1.426 (0.170) (−1.762, −1.107) 0
β_2 (Mix)	−0.581 (0.064) (−0.710, −0.456) 0	−0.596 (0.064) (−0.725, −0.478) 0	−0.574 (0.065) (−0.705, −0.451) 0
β_3 (Inc)	−	−0.177 (0.053) (−0.277, −0.070) 0.0002	−0.077 (0.076) (−0.224, 0.075) 0.1552
β_4 ($Educ$)	−0.181 (0.050) (−0.281, −0.086) 0.0002	−	−0.133(0.070) (−0.269, 0.005) 0.0302
β_5 (Age)	−0.197 (0.045) (−0.282, −0.109) 0	−0.226 (0.046) (−0.315, −0.135) 0	−0.207 (0.046) (−0.297, −0.114) 0
β_6 ($Inc \times Dist$)	−	−0.027 (0.042) (−0.109, 0.057) 0.2594	−0.032 (0.069) (−0.168, 0.102) 0.3176
β_7 ($Educ \times Dist$)	−0.005 (0.046) (−0.093, 0.084) 0.463	−	0.014 (0.072) (−0.131, 0.155) 0.4258
β_8 ($Age \times Dist$)	−0.194 (0.041) (−0.275, −0.115) 0	−0.208 (0.040) (−0.289, −0.130) 0	−0.195 (0.041) (−0.275, −0.117) 0
$\alpha_{frailty}$	2.824 (1.601) (0.927, 6.963)	3.070 (1.718) (0.972, 7.551)	2.858 (1.575) (0.923, 6.763)
θ	1.953 (0.140) (1.688, 2.239)	1.932 (0.138) (1.673, 2.214)	1.945 (0.142) (1.679, 2.224)
DIC (pD)	**264.1***** (33.1)	266.6 (32.3)	267.6 (35.8)
% outliers	3.70%% (9.40%)	3.42% (9.12%)	3.70% (9.69%)

Table 4: Summaries of the posterior distributions of $\boldsymbol{\beta}, \alpha, \theta$, DIC values (with pD) and % of units (% outliers) which have posterior $p-$value less than 2.5% (5%), with census tract and town data, from different Models. Posterior means of $\boldsymbol{\beta}, \alpha, \theta$ are followed by (standard deviations) in the first row; 95% credible intervals are shown in the second row and the posterior predictive $p-$value of the regression parameters in the third one.

of their posterior mean as $\times$ symbol and median as solid point. To assess the goodness of fit and compare Models *I, II, III* we use the DIC –refer to the bottom of Table 4(a)– and the *posterior predictive model-checking* approach, both illustrated, for example, in Ntzoufras (2009), Chap. 10. Illustrating the posterior predictive model-checking in our context, let $f(y^*_{ij}|\boldsymbol{Data})$ be the marginal "posterior predictive" distribution of the population density of territorial unit j in county i, in Massachusetts:

$$f(y^*_{ij}|\boldsymbol{Data}) = \int f(y^*_{ij}|\boldsymbol{\eta})\pi(\boldsymbol{\eta}|\boldsymbol{Data})\, d\boldsymbol{\eta}$$

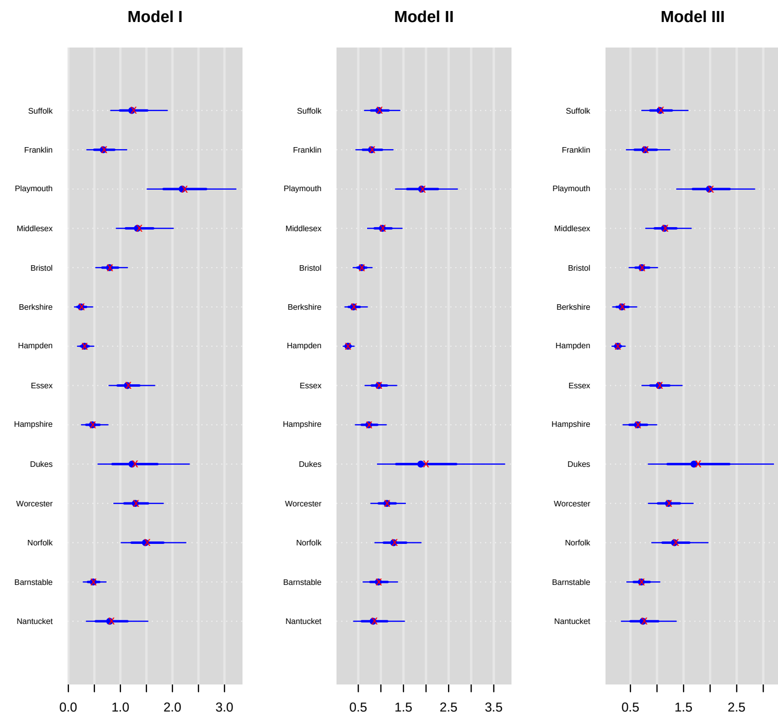
where, for simplicity, $\boldsymbol{\eta}$ denotes all the parameters: $\boldsymbol{\eta} = (\boldsymbol{\beta}, \theta, \alpha, \boldsymbol{w})$. We have simulated L draws $\boldsymbol{y}^*_{ij} = (y^*_{ij;1}, \ldots, y^*_{ij;L})$ from $f(y^*_{ij}|\boldsymbol{Data})$ for each unit in Massachusetts and we compared the actual densities of the population y_{ij} in $\boldsymbol{Data}$ with the predicted densities of the population $\boldsymbol{y}^*_{ij}$. More specifically, we summarized each marginal posterior predictive distribution $f(y^*_{ij}|\boldsymbol{Data})$ by its 95% credible interval and defined the actual values of y_{ij}'s that lie outside of the 95% credible interval as possible outliers. Indeed, if an observed y_{ij} is in the tail of the predictive distribution, then it has a very small *posterior predictive* $p-$*value*[21] (less than 2.5% or less than 5%) and this indicates a failure of the model for it. We then showed the posterior percentage (% outliers) of the units for which *Models I, II, III* fail according to $p-$value criterion in the last row in Table 4(a) as an overall measure of fit. We also graphed the posterior 95% credible interval of $f(y^*_{ij}|\boldsymbol{Data})$ as line plots in Figure 1(b) and placed the observed actual densities y_{ij} as solid dots. Finally, Figure 2(a) displays the scatterplot of *i)* the actual log population density y_{ij} (with circles), *ii)* the posterior expected census tract log densities for the reduced *Model 0* labeled by down-pointing triangles and *iii)* the posterior expected census tract log densities under (9) (with up-pointing triangles) versus the distance from Boston.

In comparing the performance of three previous models according to the DIC, at census tract level, *Model III* turns out to be the most efficient one. However, the three models perform well and similarly in identifying the outliers: indeed % outliers is equal for all the models: around 4% at 2.5% significance level and 9% at 5%. Analyzing the outliers separately in each county, we found that our Gamma-Gamma models fail in the most remote and border counties: Franklin, Berkshire and Hampshire, where the proportion of outliers ranges between 13% and 31% of the total number of each county census tracts.

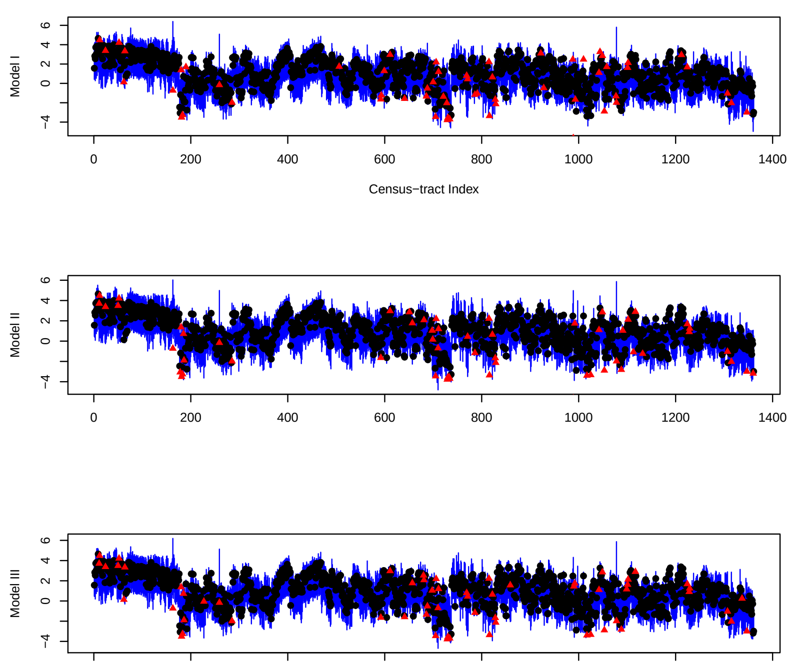
As for the regression coefficients, the three models provide consistent estimations. As we can deduce from Table 4(a), the significant terms in the prediction of density population Y are the Euclidean distance from Boston, the water land Z, and, individually, ethnic composition (Mix), income (Inc), education ($Educ$) and age (Age). The distance from Boston has a negative impact on the population density distribution ($IC(\beta_1) = (-1.233, -0.835)$). Additionally the presence of large water areas is a clear deterrent for population concentrations as expected because of the physical obstacles to construction.

Individuals also seem to be sensitive to the ethnic composition. Boston turns out to be significantly more attractive for the non-white than for the white population and, therefore, non-white individuals display stronger preferences for settling in or around Boston. This issue is generally common to other US

[21]The posterior predictive p-value is defined as the probability that an observation Y^*_{ij} governed by the posterior predictive distribution $f(y^*_{ij}|\boldsymbol{Data})$ is "*at least as extreme as*" the observed density population y_{ij}: $p^*_{ij} = \min\left\{P(Y^*_{ij} < y_{ij}), P(Y^*_{ij} > y_{ij})\right\}$

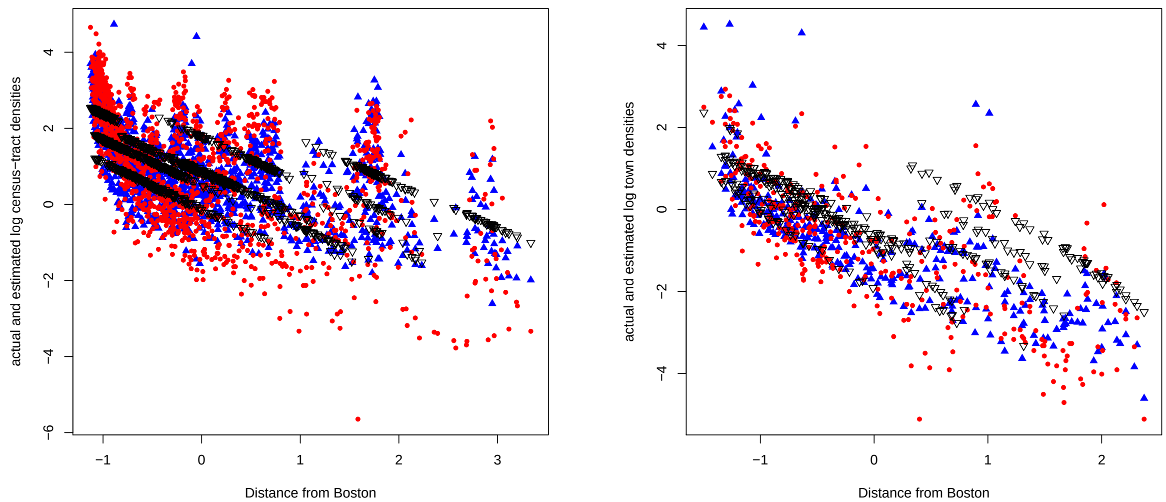


(a) Counties gamma frailties $\boldsymbol{w}$



(b) log census tracts population log densities

Figure 1: 95% posterior credible intervals of counties gamma frailties $\boldsymbol{w}$, with census tract data in Figure (a). In Figure (b): posterior predictive 95% credible interval of the log census tracts densities population with actual log census tracts densities $\ln y_{ij}$ denoted by (black) solid dots. Suspect outliers are denoted by (red) triangles up-pointing.



(a) (b)

Figure 2: Scatterplot with census tract data in Figure (a) and town data in Figure (b) (1) census tract real log densities denoted by (red) circles, (2) posterior expected census tract log densities for the reduced *Model 0* labeled by (black) triangles down-pointing and (3) posterior expected census tract log densities under *Model III* denoted by (blue) triangles up-pointing, versus distance from Boston.

metropolitan areas (McMillen 2004) and may reflect the combination of two different forces. On the one hand, non-white population (often immigrants or racial segregated citizens) usually display a stronger orientation for searching out and establishing strong social ties and seek out other individuals of their same ethnicity. Actually, in the CBD, non-white individuals are more likely to join (or interact with) other ethnic minorities, i.e. they become a part of an ethnic social network. On the other hand, the lack of a complete assimilation process between different ethnicities associated with a strong income disparities effects may shape non-white individual preferences for settling in the proximity of the metropolitan area. To this extend, the income and education indicators (as well as the interaction terms between distance and income) reinforce the previous finding: the higher one's education and income level the more willing one is to settle far from Boston. In this respect, the cohesion force of similar individuals is also captured by the age indicator. This indicator assesses that the concentration mass of the working age population is a relatively good attractor of sharing similar characteristics with other people. Instead, still in Table 4(a), irrespective of the models we fit, the interaction terms between the distance from Boston and the size of the water land Z and *Age* have regression coefficients concentrated closely around zero.

In terms of coefficient sizes, the distance is revealed to be the relative most important factor, since the magnitude of its coefficient corresponds to the sum of all of the other covariates. However the scatterplots of real and estimated log densities in Figure 2(a) infer that the only distance factor (black down-pointing triangles in the graphs) is not able to capture highest and lowest true densities (red points). The only distance *Model 0* is able to picture the trend but not the variance and, because the spread of the population density distribution, we need to address this variance. Instead, our *Model III* (blue points), which includes the proxies for subjective preferences, perform well at capturing the highest densities.

The analysis of the statistical significance of gamma frailties $\boldsymbol{w}$ confirms the importance of county random effects. First of all, the posterior means estimates of α corresponding to the different models in Table 4(a) emphasize a strong posterior evidence of high degree of heterogeneity among the counties and a posterior positive dependence between two census tracts belonging to the same county: for all *Models I, II, III* the posterior mean of α given the data is smaller than prior mean $\mathrm{E}(\alpha) = 1/0.2 = 5$, Moreover, Figure 1(a) depicts that the $\boldsymbol{w}$ values are almost all concentrated far away from one. Suffolk and Essex counties are the exception for *Models II, III*, in whose the income covariate appears. These counties are close, share borders and hence the low degree of heterogeneity may be interpreted as a signal that they would be best evaluated as a unique territorial unit.

On the whole, our results endorse the validity of our choice to introduce the concept of the subjective distance rather than the simple Euclidean distance and the Bayesian technique is revealed to be suitable for our purpose.

4.4.2 The case of a selected sample of towns in Massachusetts

In the second econometric exercise, we focus our analysis on the sample of 351 towns in Massachusetts (according to data released by the Census Bureau). We perform the exercise by running *Models 0, I, II,* and *III* as in the case of census tract data.

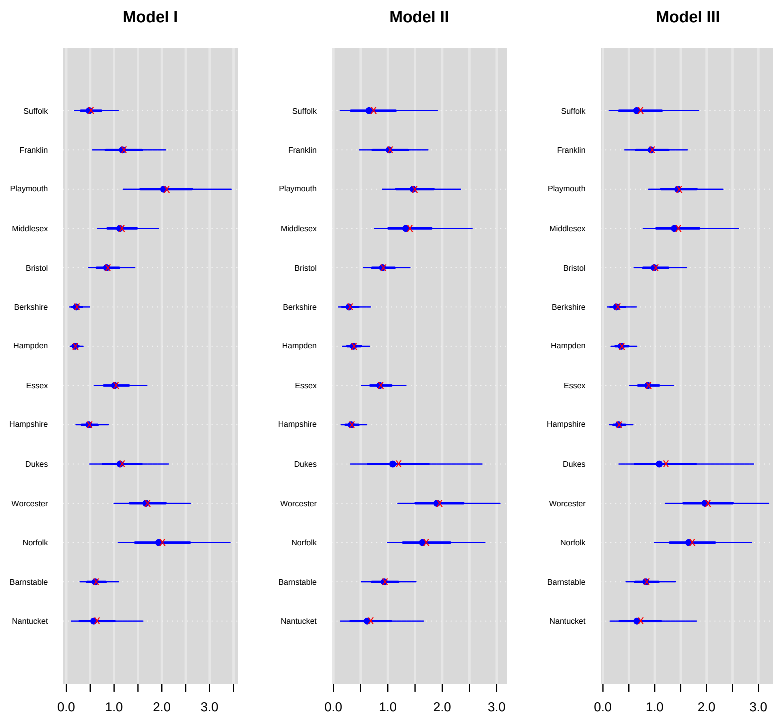
On the basis of the results presented in Table 4(b), the three specifications convey to the same results: the size of the estimated coefficients is almost identical across the three specifications as well as the statistical significance of the selected covariates. However, according to the DIC, *Model I* (including distance, ethnic composition, education and age as local covariates) is preferred. Once more the distance from Boston is negatively associated with the density size and the magnitude of the correspondent coefficient suggests that this effect is the largest among the group of covariates we consider. Looking at *Models II* and *III* we find that, again, the ethnic, education and income covariates display negative and significant coefficients confirming that high-earning, educated and white persons display preferences to settle far from Boston. Despite the results of the census tract data, in the estimations with the town data the water-area covariate is not significant at all and the age composition is shown to have a negative and statistically significant coefficient reinforcing the portrait of the selected group aiming at living far from Boston as mostly composed of working age individuals who are likely to earn a high income (rather than retired individuals). We consider the absence of statistical significance of the water areas to be mostly due to the selection criteria of the sample. We are exclusively focusing on urban territories that may be located in more or less dense urban areas in Massachusetts. Instead, this covariate is more suitable for capturing the structural heterogeneity that exists in the composition of the territorial land in urban and water areas that are latent when working with census tract data covering the entire territory of Massachusetts. The scatterplot of the real and estimated densities in Figure 2(b) for *Model I* confirms that our model (up-pointing blue triangle) provides a better fit of the true density distribution in the upper and lower tails of the distribution, whereas *Model 0* only accounting for the physical distance from Boston mostly capture the trend but without precisely fitting the observations in the tails. In addition when comparing the efficiency of the estimations of our models with the two datasets, our estimations permit a better fit of the low-density observation (in the right tails) using the town data in contrast to using the census tract data. In our opinion this outcome is mostly due to the lower degree of heterogeneity with town data that allows the covariates for a better matching with the true data. To complete the information associated with our estimates, Figure 3(a) and (b) include the graphs referring to the 95% credible intervals of the gamma frailties $\boldsymbol{w}$ and of the marginal posterior predictive distributions $f(y^*_{ij}|\textbf{town Data})$ under *Models I, II, III*, respectively. Again, the fit of our models is good because the statistics in Table 4(b) replicate the pattern described for the census tract data and the outliers (at 2.5% significance level) make up less than 3.7% of the sample for each one of *Models I, II, III*.

5 Concluding remarks

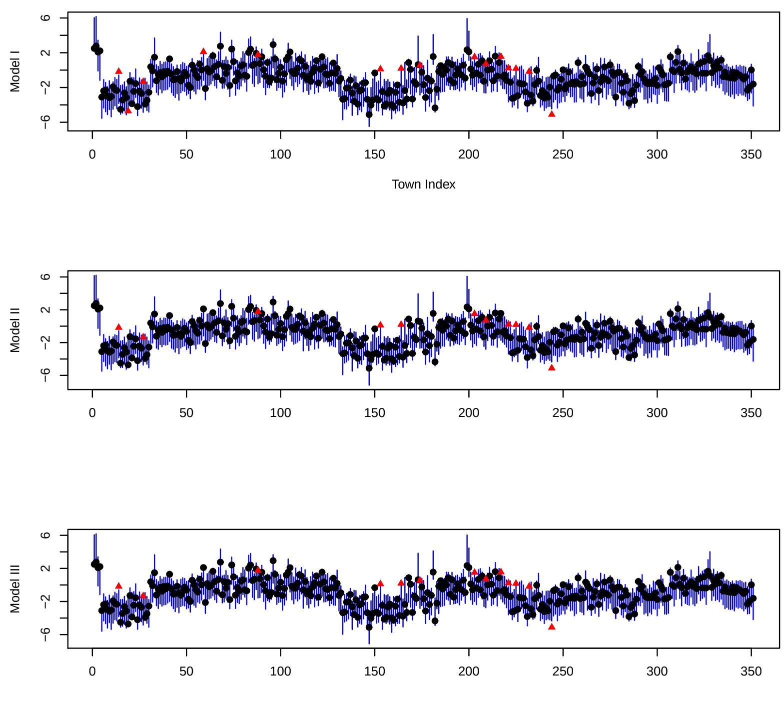
We propose a probabilistic approach to estimate the distribution density for a regional territory characterized by the coexistence of water and urban land plots. We develop a general framework by working with a probabilistic distribution built around the association of individual preferences for different location options. The population density distribution and the distance function are modeled as random variables. We adopt the Kendall's τ index to enhance the importance of the individual preferences to settle close to the center. The empirical strategy we adopt is pegged on the statistical properties of the Gamma function, which allow us to take into account the heterogeneity of the space as claimed in spatial theory. We then perform an empirical exercise to test the fit of our analysis strategy by exploiting the information of census tract and town data in Massachusetts. One novelty of this approach is the modeling of a subjective measure of distance that merges the physical distance between two locations with certain measures of individual preferences. According to the data at hand, in our exercise, the income, the age, the education and the ethnic composition jointly are as important as the physical distance from Boston for fitting the population density distribution.

In the future developments of this study, we envisage extending this empirical exercise to other case studies as well as proposing further developments of the Massachusetts case by following the historical dimension. The second idea would be to test the potential dynamic evolution of the goodness-of-fit of our framework in the context of shifting the land markets and household structure. The potential deviation, if any, from the current specification would provide some suggestions as to the interpretation of variation of habits or preferences among individuals across time.

Another natural extension is to think of a polycentric spatial configuration. In accordance with the current framework, the extension to a polycentric setting would be inspired by the Löschian central place theory to generate a proposal of urban hierarchy that overcomes the limits of a monocentric distribution. In the spirit of Lösch, the ranking of cities relies on a process of maximization of consumer welfare whereby the need to travel for any good is minimized. In this sense, we have cities that provide the highest order services (or goods) and have a huge hinterland (namely first-order places) followed by other cities (namely second-order and third-order cities) supplying progressively fewer services. The individual preferences in terms of proximity to one or more services drive the individual location decisions (e.g. to live in a first-order or second-order city). The mobility of individuals across different locations is due to their desire to access the services they demand. This movement creates potential links across the different order of towns and shapes the land structure. As a consequence of the presence of these spatial interactions, our framework should be modified by including some spatial filtering techniques such as those presented in a Bayesian framework by Crespo-Cuaresma and Feldkircher (2010).



(a) Counties gamma frailties $\boldsymbol{w}$



(b) town population log densities

Figure 3: 95% posterior credible intervals of counties gamma frailties $\boldsymbol{w}$, with town data in Figure (a). In Figure (b): posterior predictive 95% credible interval of the log census tracts densities population with actual log census tracts densities $\ln y_{ij}$ denoted by (black) solid dots. Suspect outliers are denoted by (red) triangles up-pointing.

Furthermore, the empirical results for Massachusetts suggest the need of modeling a spatial autocorrelation component between counties. Indeed, the population densities of close counties behave in a similar way. Hence, further research should more deeply characterize the basic components of the spatial structure of the territorial units.

Another important issue we did not address here is the stability over time of a Gamma-Gamma model to measure the population density in terms of the subjective distance function. Over time, the population mix can change or, occasionally, experience rapid changes, such as a sizeable immigration event or a rapid expansion of access to the internet that may well dramatically impact on an individual's perceived metric distance. As a consequence, we aim to tackle the problem from a dynamic viewpoint.

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A Empirical evidence on density population in Massachusetts; graphs and descriptive statistics

B Lemma 7

Lemma 7 If Assumption 2 is true then X, Y are discordant, i.e. $\tau < 0$.

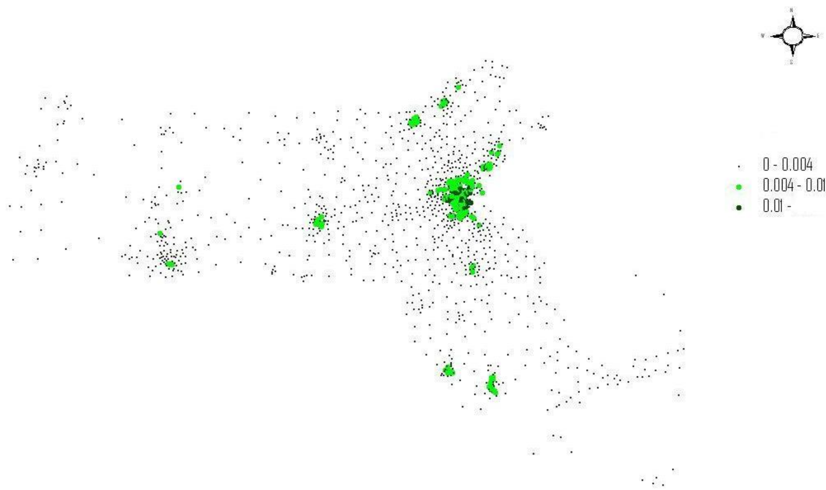

Figure 4: Distribution of population density in Massachusetts (town level) in 2000.

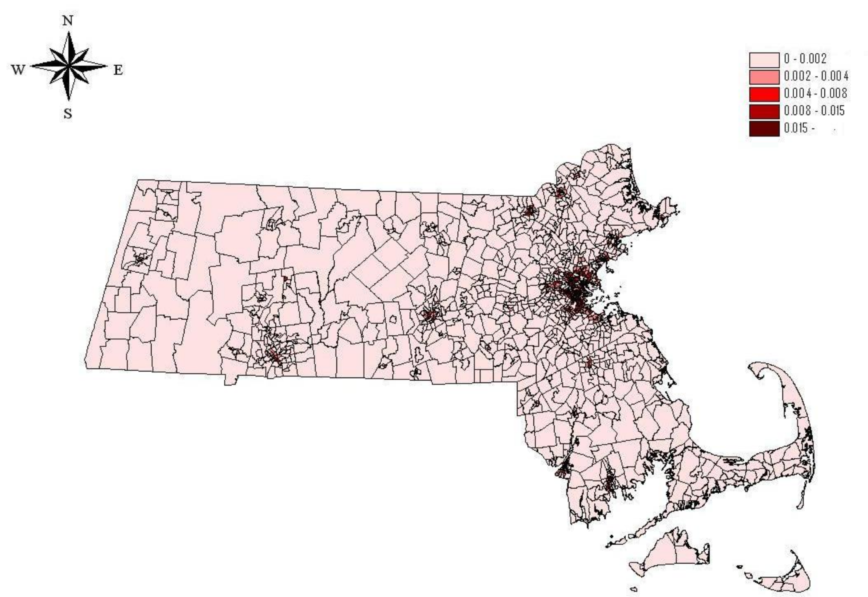

Figure 5: Distribution of population density in Massachusetts (census-tract level) in 2000.

Proof Let (X_1, Y_1) and (X_2, Y_2) be two independent copies of $(X, Y) \sim F_{Y|X} \times F_X$. Then

$$P((X_2 - X_1)(Y_2 - Y_1) < 0) = P(X_2 < X_1, Y_2 > Y_1) + P(X_2 > X_1, Y_2 < Y_1) .$$

Moreover

$$\begin{aligned}
P(X_2 < X_1, Y_2 > Y_1) &= \int_{\mathbb{R}} \int_{-\infty}^{x_1} \int_{\mathbb{R}} \int_{y_1}^{\infty} dF_{Y|X}(y_2 \mid x_2) dF_{Y|X}(y_1 \mid x_1) dF_X(x_2) dF_X(x_1) \\
&= \int_{\mathbb{R}} \int_{-\infty}^{x_1} \int_{\mathbb{R}} \left[1 - F_{Y|X}(y_1 \mid x_2)\right] dF_{Y|X}(y_1 \mid x_1) dF_X(x_2) dF_X(x_1) \\
&> \int_{\mathbb{R}} \int_{-\infty}^{x_1} \int_{\mathbb{R}} \left[1 - F_{Y|X}(y_1 \mid x_1)\right] dF_{Y|X}(y_1 \mid x_1) dF_X(x_2) dF_X(x_1) \quad \text{[by Assumption 2]} \\
&= \int_{\mathbb{R}} \int_{-\infty}^{x_1} \left(1 - \left. \frac{F^2_{Y|X}(y_1 \mid x_1)}{2} \right|_{\mathbb{R}} \right) dF_X(x_2) dF_X(x_1) \\
&= \frac{1}{2} \int_{\mathbb{R}} \int_{-\infty}^{x_1} dF_X(x_2) dF_X(x_1) = \frac{1}{2} \int_{\mathbb{R}} F_X(x_1) dF_X(x_1) = \frac{1}{2} \times \left. \frac{F^2_X(x_1)}{2} \right|_{\mathbb{R}} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} .
\end{aligned}$$

By reasoning in the same manner, we obtain $P(X_2 > X_1, Y_2 < Y_1) > \frac{1}{4}$, so that

$$\pi_d = P((X_2 - X_1)(Y_2 - Y_1) < 0) > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

and $\tau = 1 - 2\pi_d < 1 - 2 \times \frac{1}{2} = 0$. ∎

C Check of Assumption 2 in Example 6

Let $Z_1 \sim \Gamma(c, a_1)$ and $Z_2 \sim \Gamma(c, a_2)$, with $a_1 < a_2$. Then $a_1 Z_1 \sim \Gamma(c, 1)$, $a_2 Z_2 \sim \Gamma(c, 1)$ and $P(a_1 Z_1 \leq t) = P(a_2 Z_2 \leq t)$, $\forall t$. Hence

$$P(Z_1 \leq z) = P(a_1 Z_1 \leq a_1 z) = P(a_2 Z_2 \leq a_1 z) \leq P(a_2 Z_2 \leq a_2 z) = P(Z_2 \leq z) \ \forall z .$$

Consider now some conditional Gamma distribution functions $\Gamma(c, g(x_1))$ and $\Gamma(c, g(x_2))$ where $0 < x_1 < x_2$ and $g(x)$ is a positive monotone increasing function on $(0, \infty)$. Thus, $g(x_1) < g(x_2)$ and,

$$F_{\Gamma(c, g(x_1))}(y) < F_{\Gamma(c, g(x_2))}(y), \qquad \forall y > 0 \quad \text{and } x_1 < x_2 . \tag{11}$$

Applying (11) to $c = \theta$ and $g(x) = \tau \mathrm{e}^{ax}/b$ with $a > 0$, we obtain that Assumption 2 is satisfied by the Gamma model in Example 6.

D Calculation of the Kendall τ of Gamma-Gamma Model (5)

Using the model at (9) equivalent to the model at (5), we have that if $\theta = 1$, then $P(Y_{ij} > s, Y_{ih} > t)$ can be represented as the Laplace transform of a Gamma(α, α) distribution evaluated in $(s\mu_{ij}^{-1} + t\mu_{ih}^{-1})$. Indeed:

$$P(Y_{ij} > s, Y_{ih} > t) = \mathrm{E}(P(Y_{ij} > s|w_i)P(Y_{ih} > t|w_i)) = \mathrm{E}(\mathrm{e}^{-s\mu_{ij}^{-1}w_i - t\mu_{ih}^{-1}}) \ .$$

Hence $P(Y_{ij} > s) = \left[\alpha/(\alpha + s\mu_{ij}^{-1})\right]^{\alpha}$. So $P(Y_{ij} > s, Y_{ih} > t)$ has form

$$P(Y_{ij} > s, Y_{ih} > t) = (P(Y_{ij} > s)^{-1/\alpha} + P(Y_{ih} > t)^{-1/\alpha})^{-\alpha} \ .$$

The Kendall's τ of this kind of bivariate distributions has been investigated in Example 5.4 in Nelsen (1999), where one find that $\tau = \alpha/(\alpha + 2)$.

County	Obs	Mean (Std. Dev)				(Min, Max)			
		Pop	Inc	Dist	Mix	Pop	Inc	Dist	Mix
Suffolk	4	11.345 (3.573)	21.263 (5.422)	6.74 (4.42)	0.701 (0.223)	(8.001, 16.018)	(14.628, 27.374)	(0, 10)	(0.490, 0.945)
Franklin	26	0.111 (0.164)	22.419 (4.604)	145.35 (31.77)	0.974 (0.023)	(0.009, 0.836)	(12.400, 31.891)	(29, 192)	(0.888, 1.000)
Plymouth	27	0.949 (0.997)	26.643 (6.267)	47.63 (20.83)	0.964 (0.057)	(0.135, 4.392)	(17.163, 41.703)	(5, 89)	(0.687 , 0.989)
Middlesex	54	2.948 (3.943)	33.699 (11.908)	31.87 (17.03)	0.909 (0.073)	(0.120, 18.851)	(17.557, 79.640)	(5.5, 79)	(0.664, 0.985)
Bristol	20	1.114 (1.096)	24.124 (5.451)	66.11 (16.02)	0.959 (0.037)	(0.219, 0.838)	(15.602, 41.901)	(41, 92)	(0.838, 0.988)
Berkshire	32	0.143 (0.230)	25.815 (7.913)	208.19 (10.23)	0.975 (0.019)	(0.006, 1.124)	(16.381, 50.149)	(188, 231)	(0.913 ,1.000)
Hampden	23	0.814 (1.107)	22.998 (5.216)	139.91 (27.55)	0.929 (0.121)	(0.013, 4.738)	(15.232, 38.949)	(70, 182)	(0.528 ,0.989)
Essex	34	1.909 (2.290)	30.562 (8.737)	40.21 (14.07)	0.935 (0.111)	(0.231, 10.351)	(13.360, 48.846)	(14, 62)	(0.420 ,0.988)
Hampshire	20	0.306 (0.405)	22.930 (2.957)	153.25 (21.97)	0.962 (0.045)	(0.022, 1.258)	(17.427, 29.821)	(109, 192)	(0.796 , 0.997)
Dukes	7	0.204 (0.233)	24.870 (5.387)	128.14 (9.35)	0.970 (0.022)	(0.006, 0.572)	(15.265, 31.021)	(118, 144)	(0.934,1.000)
Worcester	60	0.565 (0.717)	24.424 (5.970)	77.13 (20.48)	0.950 (0.057)	(0.022, 4.597)	(16.845, 44.310)	(42, 150)	(0.741, 0.994)
Norfolk	28	1.819 (1.690)	33.576 (10.039)	31.77 (24.31)	0.922 (0.072)	(0.363, 8.410)	(23.379, 64.899)	(8.5, 143)	(0.647 ,0.984)
Barnstable	15	0.512 (0.246)	25.359 (2.194)	127.73 (27.29)	0.963 (0.022)	(0.099, 1.023)	(22.092, 29.553)	(89, 180)	(0.896 ,0.983)
Nantucket	1	0.199	31.314	112	0.887				

Table 5: Population (Pop) densities per square mile, average income (Inc) per capita (in thousand of \$) in 1999, and Proportion of white population (Mix) of each county, Source: US Bureau Census (2000), Calculus: authors. Shortest distance to Boston in km (Dist), Source:www.viamichelin.com.